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Reviewers

The individuals below have given their time and expertise to read and review manuscripts submitted for this edition of the *Banneker Banner*. We are very grateful for their help.

***Banneker Banner* Submission Guidelines**

The Banneker Banner is the official journal of the Maryland Council of Teachers of Mathematics. The journal is named after Benjamin Banneker, a Maryland native and perhaps the first documented African-American mathematician. The Banner is published once or twice each year and contains a wide range of articles on issues in mathematics education at all levels. Articles published in the journal must be submitted to the editor and undergo a peer review process.

The Banner welcomes submissions from all members of the mathematics education community, not just MCTM members. All submissions should be relevant, interesting, and useful to teachers of mathematics in Maryland. More information about the submission process can be found here.

Modeling One-Point Perspective Paintings with Analytic Geometry

Rachel Schmitz, Baltimore County Public Schools
Kristin Frank, Towson University

Abstract: *Topics in high school geometry are often studied through a synthetic approach that focuses on the construction; many geometry topics can also be studied analytically by using the coordinate plane to describe features of the geometry. We developed an activity that leverages an analytic approach – specifically systems of equations – to study intersecting lines in order to construct a mathematical model of the imbedded geometry in one-point perspective paintings. We use student work to demonstrate how this activity supports students in developing a more robust understanding of systems of equations as well a better understanding of one- point perspective.*

Introduction

Paintings that create the illusion of three-dimensional space first appeared during the Italian Renaissance in the 1400s (see figure 1) when artists started to incorporate principals of Euclidean geometry to make their paintings more realistic. For these artists painting was a form of mathematics: Leonardo da Vinci opened his *Treatise on Painting* with 'Let no one who is not a mathematician read my works' (da Vinci, 1632, as cited in Kline, 1953, p. 133). After nearly a century of experimentation, artists developed the principles of one-point perspective painting which establishes a *vanishing point* where the eye hits the canvas and all lines that are actually perpendicular to the plane of the canvas are drawn to pass through the vanishing point (see Kline 1953 for more details). These lines are called *vanishing lines* and there are typically several vanishing lines in each painting. More than five centuries after first developed, this system of realistically representing three-dimensional scenes on a two-dimensional surface forms the basis for computer visualization and movie animation (i.e., Pixar). In these contemporary applications vanishing lines and the vanishing point are represented algebraically in a digital environment.

In this paper we describe an activity where students use analytic geometry to model the imbedded geometry that enables this realistic painting style – specifically the vanishing point. Throughout this activity students engage in authentic mathematical modeling, develop their understanding of analytic geometry, and make connections between mathematics and the arts. We implemented this activity with pre-service secondary mathematics teachers, but it also aligns well with Algebra I and Geometry standards.

Hunting Scene
Spanish – artist unknown
Mid 12th century

The Adoration of the Magi
By Giotto di Bondone
1320

Saint Nicholas Providing Dowries
By Bicci di Lorenzo
1435

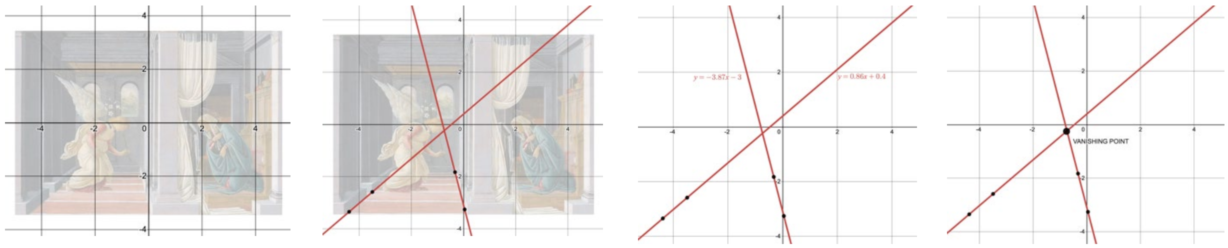
The Annunciation
By Botticelli
1492

Figure 1. It took more than a century of experimentation to develop the principles of one-point perspective: these four paintings illustrate how artists became more attentive to capturing three-dimensional space (white lines added to show lines that would meet at the vanishing point in one-point perspective).

The Activity

Each group of students selects a painting that uses one-point perspective to convey realism. The students are tasked with identifying and describing the location of the vanishing point in the painting. To help students select a painting we provided printouts of several one-point perspective paintings or students can explore online databases, such as the one hosted by the Metropolitan Museum of Art, to select their own one-point perspective painting. Next students impose a coordinate system to create linear equations to represent their vanishing lines. Since our students worked with printouts, we provided them a transparent coordinate grid overlay, dry erase markers, and rulers to develop algebraic rules for each vanishing line[1]. After students determine the algebraic rules for at least two vanishing lines they can solve a system of equations to determine the coordinates of the point intersection point of their vanishing lines – the vanishing point. Figure 2 illustrates this process. Students can verify the location of the vanishing point by constructing a third vanishing line.

[1] Students can also work in a digital graphing environment, like Desmos®, by adding the painting to the graph background and selecting points along a vanishing line. Using moveable points allows students to drag the points to the desired location in the coordinate plane.



1. Impose a coordinate system

2. Identify at least two vanishing lines

3. Mathematize the vanishing lines

4. Solve the system to determine the coordinates of the vanishing point

Figure 2. One possible modeling process to describe the location of the vanishing point.

Mathematical Modeling

The process of mathematizing vanishing lines in order to precisely describe the location of the vanishing point is an example of mathematical modeling. The Common Core State Standards (CCSS) identifies six steps in the modeling process where "choices, assumptions, and approximations are present throughout this cycle" (p. 73). These six steps are:

1. **Identify** variables in the situation and select those that represent essential features.
2. **Formulate** a model by creating and selecting an appropriate representation that describes the relationship between the variables.
3. **Compute** and perform calculations on these relationships.
4. **Interpret** the results of the mathematics in terms of the original situation.
5. **Validate** the conclusions by comparing them with the situation and improving the model, if necessary.
6. **Report** on the conclusions.

In this paper we summarize students' modeling activity as they mathematize the vanishing lines and vanishing point in a one-point perspective painting. We highlight students' work making assumptions throughout the modeling process and their work mathematizing the problem through a formulation of a model and computing relevant calculations to draw conclusions.

Making Assumptions

Mathematical modeling often involves making two types of assumptions: assumptions about the details of model and assumptions about the phenomena they are modeling. In this section we provide examples of each type of assumption.

<u>*Assumptions about the Model*</u>

One of the first decisions students made was deciding how to place the coordinate system relative to their vanishing lines and vanishing point. While Desmos® defaults to centering the image at the origin, our students were working with printed paintings and had to decide where to physically place the coordinate grid over the printout of the painting. When we implemented this activity, we found some students attempted to center the coordinate system at the vanishing point while others prioritized placing the coordinate system so that the coordinates of the points on their vanishing lines were easy to identify. This led to a productive class conversation because even though multiple groups were modeling the same painting, they chose different positions for their coordinate system and thus identified different coordinates for the vanishing point (see figure 3).

Figure 3. How students positioned the coordinate system impacted their quantification of the vanishing point in Perugino's *Delivery of the Keys to Saint Peter* (1482).

This revealed that some students assumed the vanishing point had to be at the origin (Excerpt 1).

Student 1: I was wondering, should we move the zero to the end of the point? Like this (*moves coordinate system so origin lines up with V.P.*).
Student 2: But it doesn't have to be at zero.
Student 1: We don't have to be?
Student 2: Just because it's called the vanishing point, it doesn't have to be at zero. We have to find where it is.

Excerpt 1: Students discuss assumptions about the location of the vanishing point relative to the coordinate system.

Assumptions about the Phenomenon

Students' engagement in the activity also revealed assumptions they had about vanishing-point paintings; some students assumed there are only two vanishing lines and that these lines are symmetric about some vertical axis. This is a false assumption about one-point perspective paintings; these paintings have numerous vanishing lines that are not necessarily symmetric about a vertical axis through the vanishing point. We provide two examples of how students' engagement with the paintings led them to modify their assumptions.

Early in the activity the students in Group A traced four different line segments on top of Afremov's City by the Lake[2]. They experienced difficulty deciding which line segments to select in order to identify the vanishing point because they assumed there were two specific lines that intersected at the vanishing point. We suggested the students extend the line segments they drew to see where they intersected on the page. Students were surprised to see these lines all intersected at the same point. They concluded that it did not matter which two lines they selected.

[2] Image not reproduced due to copyright.

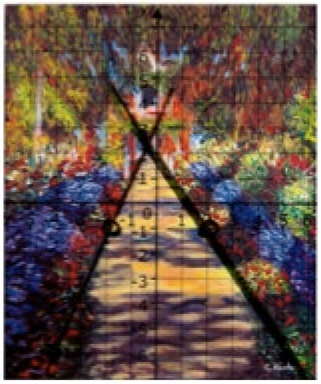


Figure 4. Group B worked on Monet's *Pathway in Monet's Garden at Giverny* (1902)

The students in Group B assumed that vanishing lines had to by symmetric about the *y*-axis and the vanishing point would lie along the *y*-axis. As they identified points on their vanishing lines, they experienced perturbation because their lines were not symmetric about the *y*-axis (see Excerpt 2). These students were convinced they did something wrong and were not able to proceed with the activity. We intervened and asked if they could identify other vanishing lines in the painting. Identifying a third line that they did not expect to be symmetric resolved their internal conflict.

> Student 3: I think they should be symmetric. Like, when you fold it (*folds coordinate system along vertical axis*) it should be the same line.
>
> Student 4: (*Labels four points on vanishing lines*) This is not right. This one is 5 and this one is -4. This should not be! I get $^{5}/_{2}$ and I think it should be $^{-5}/_{2}$ but it is $^{-5}/_{3}$.

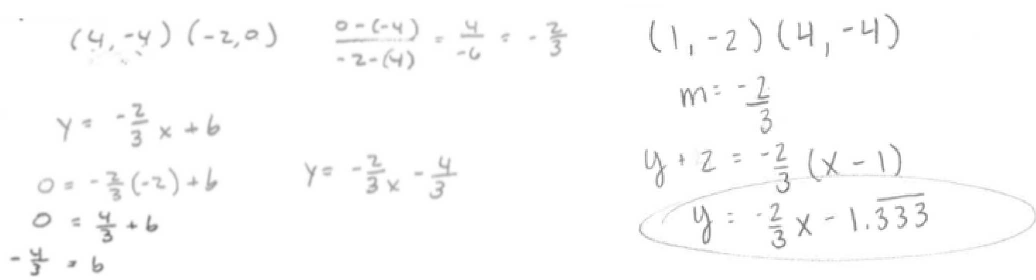
Excerpt 2: Two students discuss how the two vanishing lines should be symmetric and what it would mean for the lines to be symmetric.

Students' difficulty with constructing and selecting vanishing lines reveals how the students' understanding of the phenomena, in this case technique of vanishing point perspective, impacts the modeling process and specifically the assumptions one makes during the modeling process.

Compute: Mathematics to Draw Conclusions

After students drew geometric representations of selected vanishing lines, they worked to algebraically describe these lines to determine the precise location of the vanishing point. This is part of the *compute* phase of the modeling process since students leveraged their graphical representation to perform calculations that allowed them to draw conclusions. Authentic modeling tasks will have more than one way to formulate the problems and organize the mathematical computation. In this activity students made decisions about the algebraic forms to use and the method to solve the system of equations.

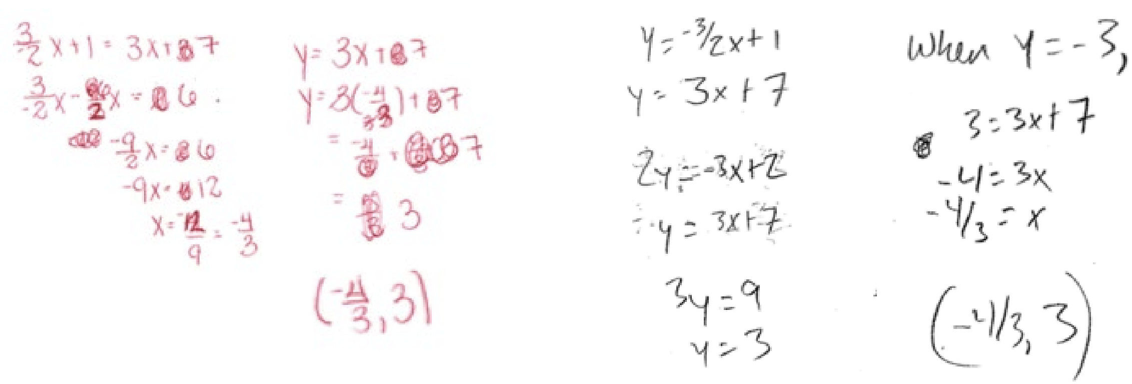
Two students in Group C were working to identify the equation of a vanishing line. Student 5 selected the points (4,-4) and (-2,0) along the vanishing line and used slope-intercept form to determine the equation of the line. Student 6 selected the points (4, -4) and (1, -2) and used point-slope form to determine the equation of the line. Both students determined the same algebraic rule despite using different methods to construct the rule (See figure 5). These students were initially surprised that they ended up with the same answer despite their different methods. However, they concluded that since they were modeling the same vanishing line the mathematical representation of that line should be the same. Here a vanishing line provided a context for the students to understand mathematical equivalence.



Student 5’s work Student 6’s work

Figure 5. Two students from Group C used different coordinates and different algebraic methods to determine the equation of a vanishing line.

Just as students within Group C used different methods to develop their linear equations, students in Group D used different methods to solve their system of equations (see figure 6). During a class discussion of this work students compared solution strategies to recognize that both substitution and elimination gave the same intersection point – the same coordinates for the vanishing point.



Student 7’s work Student 8’s work

Figure 6. Two students from Group D used different methods to solve the system of equations.

These examples reveal how authentic modeling tasks can support students in understanding that a problem does not dictate a method of solving but instead it is the mathematician’s choice of what solution method or skill to use in order to help them solve the problem. These types of modeling tasks can help students focus on the process of doing the mathematics and mathematical equivalence as opposed to a particular solution method or answer.

Conclusions

When designing modeling activities, it is important that the task is left open-ended so that students can make assumptions, question the validity of the assumption, and have a choice in the mathematics to use when constructing the model. In this activity we did not give participants specific instructions on how to model vanishing lines and the vanishing point. As a result, students made assumptions about the modeling process (where to place the coordinate system) and confronted their assumptions about the relationship between vanishing lines in a vanishing point perspective painting. Participants also demonstrated different ways to mathematize their vanishing lines, including how they determined the equations of lines and how they solved the system of equations.

We believe that the students' different approaches aided the learning process. By providing students with an opportunity to engage in mathematical modeling they learned about the characteristics of vanishing point perspective paintings and made connections between various algebraic methods for constructing linear equations and solving systems of equations.

Acknowledgements

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Scaffolding Problem-Solving

Mark A. Shore, University of the District of Columbia
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Abstract: *This paper proposes the use of scaffolded problems in building both preservice elementary teachers' and elementary school students' problem-solving ability. It shows problems developed or modified by the author that require problem-solving and are scaffolded to build understanding. It describes the experience of a professor that attempted to teach a difficult applied compound proportion problem in a Math for Elementary Education course. The same problem was then adapted to a series of problems that built students' reasoning to the level needed to solve the initial problem. A second problem on systems of equations is also presented in a scaffolded manner. While these problems may seem beyond the ability of elementary school students, scaffolding gives students the opportunity to build their understanding, lowers frustration for both the teacher and the students, and increases productive persistence. A discussion on the use of scaffolding in other content areas is also described.*

Introduction

At many colleges, the Math for Elementary Education course(s) is a specific math course offered in either the Mathematics or Education Departments and covers topics in methods of teaching elementary school mathematics, while reinforcing the preservice teacher's math skills. Problem-solving is a topic in this course and George Polya's four-step problem solving approach is discussed and used. One problem that students of all ages have is step two of the problem-solving process, "Devise a Plan." Teaching problem-solving skills is difficult and can be a frustrating process for both the students and the professor (Liljedahl, 2021). Kusumadewi and Retnawati (2019) found very low problem-solving skills for preservice teachers. Studies have shown that the proportion of textbook tasks requiring more than using known procedures is low (Brehmer, et. al., 2016; Lithner, 2004; Schmidt, et. al., 2012). As several scholars have noted, problems are central to mathematics teaching and learning and constitute the basis for intellectual activity in the classroom (Lampert, 2001; Stein, et. al., 2007). In Peter Liljedahl's book, "Building Thinking Classrooms in Mathematics", he describes horrible teaching situations of teachers trying to get students to solve problem-solving questions when the problem is given to students with no scaffolding.

Problem 1 without scaffolding: If 6 cats can kill 6 rats in 6 minutes, how many cats are needed to kill 100 rats in 50 minutes? This is a famous problem often attributed to Lewis Carroll but it originated earlier in 1879. When we think of the period in which this problem was written, getting cats to kill rats was probably an issue that people could relate to. The instructor for the Math for

Elementary Education course originally tried to have the students solve this problem without any scaffolding.

In trying to promote a “Thinking Classroom” in the Math for Elementary Education course, the instructor allowed the students time to try to solve the problem. After a few minutes, the teacher asked the students to explain their reasoning in trying to solve the problem. One student said they were trying to set up an equation but could not come up with it. Another student tried to reason their way through the problem, saying, “If 6 cats can kill 6 rats in 6 minutes, then 1 cat can kill 1 rat in 1 minute.” The professor replied, “If 1 cat can kill 1 rat in 1 minute, then wouldn’t 6 cats be able to kill 6 rats in 1 minute?” This contradicts the premise of the problem, that it takes 6 minutes for 6 cats to kill 6 rats. After more discussion and with help from the instructor, the class got to the point that 1 cat could kill 1 rat in 6 minutes. The instructor told the students to determine if a variable is working for the cats or against the cats. For example, if the time remains the same and we increase the number of rats, will we need more cats? If the number of rats remains the same and we decrease the time, will we need more cats? The students were completely frustrated and work by the students ended.

The professor realized that the students were unable to solve this problem in class. The discussion then went to the difficulty of the problem and how to make it more manageable for students. The class jumped into a difficult problem-solving task with no practice using a similar less complicated problem. Could we scaffold the problem-solving skills in this problem so that students could build to this level of difficulty? The discussion then went to the appropriateness of the problem in the classroom. Could we have this same concept (compound proportionality) taught using problem-solving with a modern topic? In addition, the “cats and rats” problem can be Googled to find the solution with the reasoning behind the solution. With the help of the students, new problems were written for homework that use scaffolding and a topic that may be more interesting to elementary school students. The questions below were written on My Open Math in a specific order to build student reasoning from first working with direct and indirect variation to the level needed to solve compound variation problems. Student answers to each question are automatically graded so that students get immediate feedback. In addition, random variables were used in creating the problem on My Open Math, so that each student got a similar problem with different values.

New Problem 1 with scaffolding: Using the fact that 4 kids can assemble 8 bikes in 2 hours, answer the following.

Part 1: Keep the number of bikes constant at 8 bikes. What is happening to "kids" and "time"?
1. How many kids would it take to assemble 8 bikes in 1 hour?
2. How many kids would it take to assemble 8 bikes in 30 minutes?
3. How many kids would it take to assemble 8 bikes in 4 hours?
4. To assemble the same number of bikes in 1/3 of the time, it would take _____ times the number of kids.
5. To assemble the same number of bikes in 5 times the amount of time, it would take _____ the number of kids.
6. Use the information given in the problem and your previous answers to write an equation using *b* for bikes, *k* for kids and *t* for time (in units of hours).

Part 2: Keep the amount of time constant at 2 hours. What is happening with "kids" and "bikes"?

1. How many kids would it take to assemble 16 bikes in 2 hours?
2. How many kids would it take to assemble 4 bikes in 2 hours?
3. If you want to assemble 5 times the number of bikes in 2 hours then you will need ______ times the number of kids.

Part 3: More than one variable changing.

1. How many kids would it take to assemble 16 bikes in 1 hour?
2. How many kids would it take to assemble 24 bikes in 30 minutes?

Part 4: Use what you learned to try a new problem.
Use the fact that 6 kids can assemble 6 bikes in 6 hours to answer the following.

1. How many kids would it take to assemble 24 bikes in 12 hours?
2. How many kids would it take to assemble 100 bikes in 50 hours?
3. Use the information given in the problem and your previous answers to write an equation using *b* for bikes, *k* for kids and *t* for time (in units of hours).

In going through the problems, students build their reasoning to better understand how to solve more difficult problems. The final problem has the exact same values as the "cats and rats" problem, but the scaffolding gives the students a reasoning building pathway to get to the solution.

Problem 2 with scaffolding: You are a zoologist on an alien planet. One animal called the triclops has 3 legs, and another animal called the pentalope has 5 legs. You did your best to count both the number of legs and the number of animals. You want to check to see if you counted correctly.

1. If you counted a total of 13 legs, how many of each type of animal did you see?
2. If you counted 30 legs, how many of each type of animal did you see?
3. If you counted 50 legs, how many of each type of animal did you see?
4. Do you see a pattern in your answers for the previous problem? What was the pattern?
5. If you counted 74 legs, how many of each type of animal did you see?
6. If the total number of animals was 20 and the total number of legs was 74, how many of each type of animal did you see?
7. If the total number of animals was 20 and the total number of legs was 58, how many of each type of animal did you see?
8. If the total number of animals was 20 and the total number of legs was 102, how many of each type of animal did you see?
9. If the total number of animals was 20 and the total number of legs was 71, how many of each type of animal did you see?

This problem scaffolds the problem-solving skills from problems that only have one constraint and one answer to problems that have multiple answers or multiple constraints. Most students answered this problem by making a table like the one below.

Number of animals	Number of legs from triclops (*t*)	Number of legs from pentalope (*p*)
0	0	0
1	3	5
2	6	10

The students continued the table as needed. The solutions are listed as ordered pairs in the form (T, P). Part 1 only has one solution (1,2). Part 2 has two solutions: (0, 6) and (5, 3).

Part 3 has four solutions: (0, 10), (5, 7), (10, 4) and (15, 1). For Part 4 of the problem, students saw the pattern, that they got a solution at every five triclops or every three pentalopes. They realized that once they were able to get one solution, they either decreased or increased the number of triclops by five and did the opposite operation (decrease or increase) to the number of pentalopes by three. Part 5 has five solutions: (3, 13), (8, 10) and continues with the pattern of ($3+5n$, $13-3n$) until the point (23, 1). Part 6 of the problem is a typical applied systems of equations problem and is where the problem would have started without scaffolding. However, with scaffolding elementary school students are able to answer the problem. Part 6 has one solution (13, 7) and students can use part 5 to find the correct answer. Part 7 has no solution, since the number of legs is less than the least number of legs possible for 20 animals. Part 8 has no solution, since the number of legs is greater than the largest number of legs possible for 20 animals. Part 9 has no solution since you can't get 71 legs with 20 animals. Part 9 shows the additional requirement for whole-number answers.

Summary/Discussion

Students will see problems like Problem 1 when they learn to solve problems involving direct, inverse, and combined variation. When students are formally exposed to different types of proportions in Algebra, they build up to compound proportion problems. In the same manner, Problem 1 builds students' understanding by working up to compound proportions. Students will see problems like Problem 2 when they get to the study of systems of equations. However, Problem 2 goes beyond the typical systems of equations problem by requiring whole number solutions. Seeing these types of problems and using a scaffolding process, builds students' problem-solving skills in elementary school and provides a link to Algebra.

Additional Applications of Scaffolding

Scaffolded teaching is not limited to elementary school students. Professors should be aware of the principles and strategies of good *problem-solving* within their discipline. They may have students identify specific problems, difficulties, or confusions that exist in the discipline. If the students are unable to articulate their concerns, the professor should determine where they are having trouble understanding the concept by asking them questions. Finally, have the students articulate their problem-solving process.

As students work to establish their own problem-solving skills have students work through a variety of problems on their own. Be careful to only offer assistance to overcome obstacles. Since group work is common in many college courses, allow students to talk through the process. Students can often support each other when talking through a problem. This will help them reason more sympathetically about the steps needed to solve the problem. In addition, group work assists students in realizing that problems often have numerous solution strategies, with some being more effective than others.

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JoAnna Burley Shore has taught in the Department of Management at Frostburg State University for 24 years. Dr. Shore's interest in problem-solving has led her to introduce it into her graduate and undergraduate courses. She feels strongly that all disciplines require expertise in problem-solving regardless of whether it is accounting, finance, marketing, or management.

Preparing for the Unexpected: Coaching Teachers to Use Responsive Questioning

Edward C. Nolan, Moravian University

Abstract: *Coaches can advocate for teachers to plan and teach ambitiously. This can include providing planning experiences where teachers and coaches collaboratively solve tasks, anticipate questions to uncover evidence of student thinking, and discuss and evaluate possible lesson pathways and teacher actions. These experiences allow for the exploration of unexpected student responses and ways to both engage and enhance student thinking. This article explores ways in which coaches can support teachers during co-planning sessions in the use of ambitious teaching techniques involving comparing ratios, from designing an effective task to asking questions that help students to build on their own thinking.*

When teachers create a lesson plan, they are providing a glimpse into how they envision the progress of a lesson. Sometimes the lesson plays out in ways similar to this vision, and sometimes the lesson goes differently than anticipated. Ambitious teaching (Lampert et al., 2013) requires teachers to not only use the preconceived ideas developed in their lesson plans but also to incorporate in-the-moment decisions to integrate the focus both on the learning goal and the thinking of the students. This article explores how coaches can support teachers to engage in ambitious teaching by planning questioning that allow students to retain the power to solve a given task.

Lesson plans detail what teachers expect to happen in their lesson (Cunningham, 2009). Teachers should plan questions they think they might ask to support students to engage in a task, create possible answers to those questions, and develop multiple pathways students may take to solve the task. These answers and pathways represent what teachers *think* will happen. But sometimes student ideas are different than what is anticipated, or students need appropriate scaffolding to achieve the learning goal. These variances result in a need to balance the use of pre-planned questions with on-the-spot actions based on what is observed. Planning time with a coach can model different pathways and consider how anticipated questions can blend with evidence of student learning to best meet the needs of the students. How can coaches support teachers to balance what they expected from students in their planning with what is provided by students while enacting the lesson to achieve maximum student understanding of the learning goal? And how can coaches support teachers to develop a decision-making process consistent with ambitious teaching that promotes the use of effective, and responsive, questions?

The TQE Process

By integrating research and experiences in supporting both pre-service and in-service teachers, this article explores using the TQE Process to plan and implement questioning – both in how teachers ask questions and how they use student responses as evidence of what students are thinking. The TQE Process involves selecting a **task** aligned to a specific learning goal, planning and asking **questions** that support mathematical processes and practices, and collecting **evidence** of student thinking and understanding of the learning goal (Dixon, Nolan, & Adams, 2017). This process ties closely to important mathematical teaching practices, including implementing tasks that promote reasoning and problem solving, posing purposeful questions, and eliciting and using evidence of student thinking (National Council of Teachers of Mathematics, 2014). Coaches can support teachers through development of lesson plans and modeling in coaching sessions (a sample agenda is provided in figure 1).

Identify the Learning Goal – 10 minutes
- Link to Standard(s)
- Determine possible prior knowledge

Create/Adapt Task – 20 minutes
- Write or adapt a task tied to the learning goal
- Solve the task in as many ways as possible
- Share solutions and discuss possible common errors

Plan questioning – 10 minutes
- Anticipate correct and incorrect responses
- Questions to assess understanding and advance learning

Modeling/Practice – 20 minutes
- Act out possible scenarios

Figure 1. Sample coaching session agenda

This method of planning begins with making sense of the learning goal.

Making Sense of the Learning Goal and Planning the Task

As a coach, you may have the opportunity to support teachers by co-planning lessons. Lesson planning starts by clearly identifying a learning goal. This is more than selecting an appropriate standard, as the learning goal needs to target what is planned to be accomplished within the scope of an individual lesson. For example, consider the standard of using ratio reasoning to solve problems (typically part of the sixth-grade curriculum). It is important as a coach to ask teachers how they would break down the instruction of this standard and determine the lessons needed to achieve the standard, as it is not likely to be accomplished in one day. Of course, planning lessons does not necessarily mean that each and every student will be proficient by the end of each lesson on each learning goal, as some students may reach proficiency at different times in the unit. For ratio reasoning, lessons should be developed that afford students opportunities to work with different representations of ratios, equivalent ratios, and comparing ratios. For the purpose of this discussion which focuses on the development of questioning, imagine that students have already explored representing ratios and determining equivalent ratios with some success and the learning goal for the next lesson is to compare ratios using a real-world context to solve a problem.

A **task** needs to be selected, adapted, or created that helps students make sense of the learning goal. Let's consider the orange drink task (see figure 2).

> Arianna is making an orange drink for her birthday party. She has decided to try two different mixtures. For the first one, Arianna mixes 3 quarts of orange juice with 4 quarts of ginger ale. For the second one, she mixes 5 quarts of orange juice with 6 quarts of ginger ale. Which drink will have the stronger orange taste? How do you know? (Nolan, Dixon, Roy, & Andreasen, 2016, p. 2)

Figure 2. Orange drink task

The next step in planning is to solve the task, looking for multiple solution pathways, both correct and incorrect, that students may use in solving the task. This provides insight into the types of questions and evidence of student learning that teachers expect in the lesson. Solving tasks during planning limits the number of in-the-moment decisions required during instruction. This is especially helpful for teachers new to the grade level, course, or profession.

One model that students might use involves equivalent ratios to show the same value for one of the quantities (see figure 3).

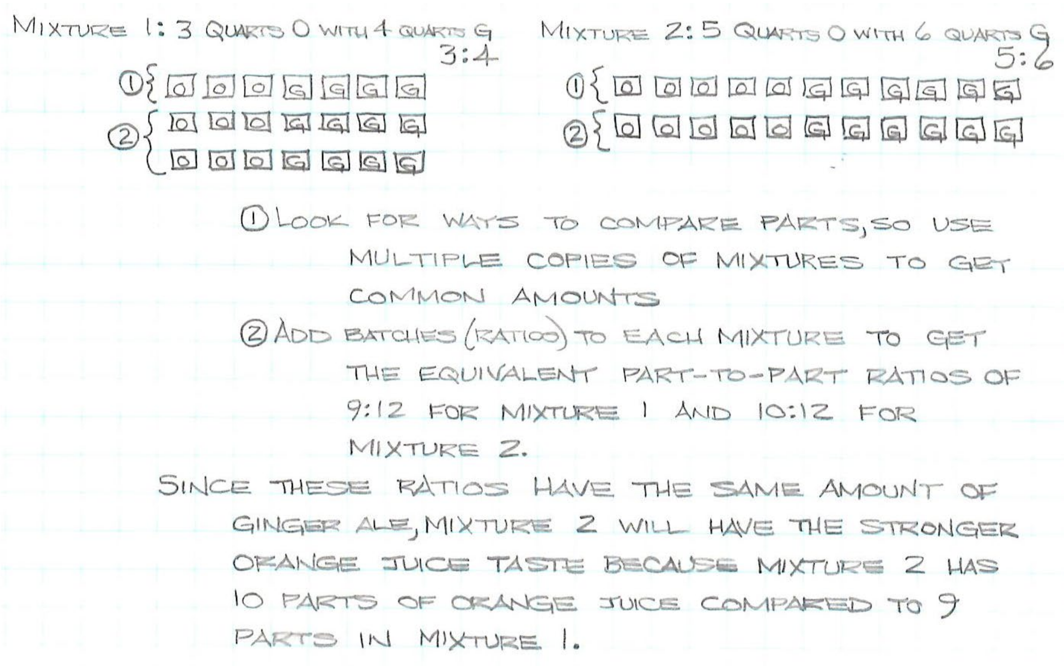

Figure 3. Possible teacher-created solution strategy for the Orange Drink task

Here, each mixture is modeled as it is represented in the task: mixture 1 as 3 parts O (for orange juice) to 4 parts G (for ginger ale) and mixture 2 as 5 parts O to 6 parts G. In this solution, it is reasoned that since both 4 and 6 are factors of 12, the orange drink can be described by using 12 parts for both mixtures. In this way, the two ratios of the quantities can be compared. Representations for parts are added in the same ratio to both mixtures (two more of the ratio 3:4 for mixture 1 and one more for the ratio 5:6 for mixture 2) to have 12 parts of ginger ale in each mixture. This shows that the new representations of the two mixtures have the same number of quarts of ginger ale. The two ratios can now be represented as 9:12 and 10:12. Comparing the two mixtures can be done by demonstrating which mixture has more orange juice to the same quantity of ginger ale. Mixture 2 has more orange juice compared to the same quantity of ginger ale, so it has the stronger taste. This is one strategy that students may use as they are connecting their developing knowledge of ratios to compare the two mixtures.

The most effective tasks uncover common errors and provide students many different solution pathways. The choice of values is critical in the design of a task, as the number choice allows for students to exhibit their thinking and understanding of the learning goal, both correctly and incorrectly. In this task, the values are chosen to provide students different paths to find relationships between the quantities as well as uncover possible developing conceptions. One incorrect interpretation would be for students to think that both mixtures will be the same strength of orange taste, because mixture 2 begins with mixture 1 but adds two quarts each to the orange juice and ginger ale. When tasks are linked to the learning goal and designed to uncover errors, teachers can see how students are reasoning and use student thinking in the development of understanding, linking the task to the collecting and use of student thinking in the TQE Process.

Planning for Questioning and Collecting Evidence of Student Thinking
In the next step of planning, teachers anticipate what may happen in the lesson, creating a "hypothetical learning trajectory" (Simon, 1995, p. 135) for the lesson. Planning should include questions, predictions of possible student responses, and actions teachers may take based on these student responses. As a coach, it is important for teachers to plan these questions yet know that questions asked in the lesson may be different as they need to encourage students to make sense of the task in their own way. Teachers need to balance their use of questions to guide students toward the learning goal through eliciting and using student thinking.

Coaches should help teachers develop possible **questions** that can press for student explanation (if needed) or can be used to help students link – or prepare to link – to solution pathways of other students (Boston, Candela, & Dixon, 2019) and lead to meaningful conversations in class. High-cognitive-level questions require students to think conceptually and lead students to make connections (Stein & Smith, 1998) and can be challenging to create in the context of instruction (Nolan, 2019). These questions may also include extensions of a task once a student has demonstrated understanding of the learning goal. Some teachers may feel that this is challenging to imagine, but effort put forward in planning can provide a source of high-cognitive-level questions that may be difficult to create while teaching. This is why planning is so important to effective questioning (Nolan, et al., 2016).

Just as teachers want to encourage students to do the thinking and develop their own solutions, coaches need to ask questions that will help teachers to determine actions that will support their students in meaningful ways. The first consideration for teachers is to be sure that students understand the task and how they see the strategy solving the problem. There are basic teacher questions such as, "Why did you use this representation?" or "How does your solution connect to the task?" While these questions can be helpful during instruction and are worth reminding teachers to use, they are generic and apply to most tasks. For planning, it is helpful for teachers to spend their valuable planning time to develop questions specific to the task. As a coach, asking teachers to think about questions and expected student responses for a task is an important element in effective planning.

Questions a coach can use at this stage include:

1. What about the task might "cue" students to use a particular strategy? How can you (the teacher) help students identify those elements without "telling" them what to do?
2. How will you help students connect the information in the problem to a possible strategy without giving away the strategy?
3. What will **evidence** of student thinking look like with this task? What will you do with what you learn to help students develop their strategy?
4. How will you extend the thinking of students who complete the task quickly and can explain their thinking well?

Supporting Student Thinking Through Questioning

It is important to help teachers create student-centered learning environments that focus on the thinking of the students. Too often teachers ask questions that give students a strategy or guide students to a particular strategy, usually the one that teachers have anticipated, rather than building on the thinking of the student (Nolan, 2019). This is the difference between "funneling" and "focusing" student thinking. In funneling, teachers guide students to a planned strategy, providing the reasoning and thinking for the students (Wood, 1998). For example, consider the teacher asking the following question as an opening support to a student, "Is there a way for one of the quantities in your problem to be the same?" Here the teacher is guiding the student to view the task in the way that the teacher views it. While this can be helpful when working to support the development of an algorithm, such as the long division algorithm, it may restrict student thinking in more discovery-based tasks. Rather, when teachers ask questions about the mathematics of a student's solution to help the student uncover important mathematics within the solution, they are focusing on using student ideas to support the development of mathematical understanding (Wood, 1998).

"Cueing" Students from the Task

Teachers need to consider how to support students to interpret tasks and how students identify information in a task to develop their own strategies. As a coach, you can help teachers to do this in ways that will maintain the student-centered learning environment. In the Orange Drink task (see figure 2), how does the teacher encourage students to make sense of the task without "giving away" strategies? All too often, teachers begin with a high-cognitive-demand task and then lower its cognitive demand during implementation (Boston & Smith, 2009). Ask teachers how they would engage a student who is not sure where to start. Teachers should respond in ways that do not lower the demand of the task. As part of planning, teachers should think about questions to ask students in order for students to determine what is important in the task and what about the task they need to know. This information about the task should not come from the teacher but rather should come from students' responses to prompts from the teacher. Initial questions could include:

- "What do you know from the task?"
- "What do you need to do?"
- "How could you represent the situation?"

Note that the questions do not provide a specific direction to students. They are generic and could be used on many different tasks. They are intentionally vague. In this way, students still have the opportunity to continue on a path based their own strategy. It is important for teachers to understand the reasoning behind using vague questions at this stage of the lesson (as questions will narrow later in the discussion). Teachers can sometimes lead students to use certain strategies by guiding students to see relationships that the teacher is aware of but that the student may not yet see. For example, a question like, "What can you tell me about the relationship between the two mixtures?" may guide students to look at a relationship between mixtures first rather than looking at the mixtures individually. Helping teachers to encourage students to think about problems in ways that the students, themselves, generate ideas is an important step in maintaining a student-centered learning environment.

Tying Questions to Strategies

In the planning stage, coaches may establish a scenario in which teachers see the beginnings of their own strategy with the student representing the ratios 3:4 and 5:6 in a similar manner to what teachers anticipate, but the student appears to be stuck (see figure 4).

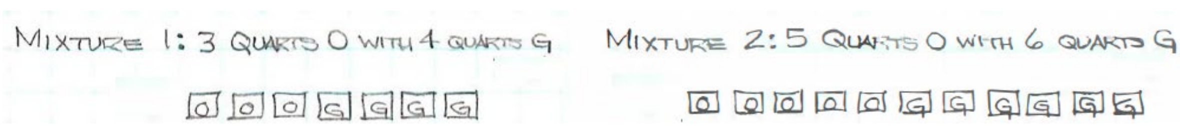

Figure 4. Possible representation of the task

Once the scenario is shared, ask teachers what they would do to support the student. Here, you, as the coach, help teachers create and evaluate different responses that they might offer during instruction. What follows are anticipated questions along with possible student responses.

Possible Response 1: The teacher asks the student, "How do you plan to use the information you have here?" With this question, the teacher is probing for additional information about the thinking of the student. There is no assumption about the direction the student is considering. In fact, the teacher is seeking more information about the direction the student is considering. The next action would be based on how the student responds.

- If the student says they are not sure, then the teacher will need to ask questions that help guide the student to explain, such as "What do you know about the ratio of 3:4?" or "How would you compare what you have created?" Some students will be unsure of sharing their thinking. It is important for students to share their thinking whether correct or incorrect and learn that early stages of thinking will develop into effective strategies as long as they are shared and explored. Probing questions may be enough, or more guiding questions may be needed if the student continues to struggle. Guiding questions could ask students how they could compare ratios or how they would compare fractions in a similar scenario. This is how the teacher is using the progression from general questions to specific questions to learn about student thinking without providing direction to the student too early.
- If the student provides a mathematically incorrect response, such as, "I see that the second ratio is the first ratio with two cups of orange juice and two cups of ginger ale added, so both mixtures will taste the same," the teacher could move to a discussion

involving the first ratio. The teacher could ask, “If you add three quarts of orange juice to mixture 1, how much ginger ale should you add?” The goal is for the student to understand that increasing the size of the parts needs to be done in relationship to the ratio and not in equal amounts (unless the ratio is 1:1). Note: This student response is a common error in comparing ratios and should be explored as part of the lesson. If this idea that the two ratios are equivalent because of adding the same to both quantities is not shared by students, it should be offered as a suggestion and discussed as a class. It is important that students experience common errors during instruction and learn how to use mathematical reasoning to demonstrate the error. As discussed earlier, this is an important part of the selection of the quantities included in the task.

- If the student states a mathematically correct response, then the teacher should provide a non-evaluative response, such as “interesting” or “ok,” ask the student questions that provide for their reasoning and allow the student to continue working. It is important at this stage that the teacher does not provide evaluative feedback on either correct or incorrect responses, as the thinking and determination of the correct pathway to a solution need to be retained by the student. It is also important that the teacher ask for reasoning for both correct and incorrect responses, as not asking about correct responses will provide evaluative feedback to students over time. Delaying feedback here encourages student engagement in the thinking process, building their autonomy (Shute, 2008).

Possible Response 2: The teacher asks the student, “What would happen if you were to draw more representations of the same ratio for both sides?”

- While not an inappropriate action, it does guide the thinking of the student. It would be better to ask a question that allows for the student to provide the evidence of their thinking. This question may be helpful after the student has shared their thinking or if the student is unsure of how the current representation can be used to answer the question.

Possible Response 3: The teacher asks the student, “How could you get the same number of parts for one of the quantities” or “How could you get the same number of parts of ginger ale?”

- Here, the teacher is funneling the student to the teacher’s view of the problem. As the student has started their strategy in a way that the teacher recognizes, this question guides students along a trajectory that the teacher has anticipated. While guiding the student towards a successful strategy, it is not clear that this is what the student was thinking and the “leap” to this step of the teacher’s thinking may not be understood by the student. For example, the student may be thinking about the possibility of adding the same quantity to both sides or merely comparing the quantities of orange juice without using the information about ginger ale. It is important for the teacher to probe and allow the student to guide the direction on the discussion.

During the planning discussion of the scenario, multiple questions and pathways should be explored, even if teachers do not provide them, so that teachers see that there are many options available, and that each choice provides an avenue to either a more student-centered or less student-centered learning environment. As a coach, providing different possible scenarios supports the development of teachers and encourages them to plan for and implement tasks with multiple solution pathways. Understanding the importance of listening to the thinking of the student, anticipating how to address the evidence the student provides, and selecting options that maintain the thinking with the student, is critical in planning and implementing lessons effectively. In planning for multiple options and addressing each of them, you are modeling the same process that teachers should be using in their classrooms – using multiple responses to develop student understanding with depth. It is often helpful to share this with teachers so that they can understand the approach both in their planning and their teaching.

What other strategies could students demonstrate? Teachers should try to anticipate many different strategies students might use and consider what actions to take if students start in one direction and appear to need help, as described above. This is a place where it is an advantage for teachers to plan collaboratively, as others may think of strategies to solve the task that one teacher might not create.

Another possible strategy for this task is presented in figure 5.

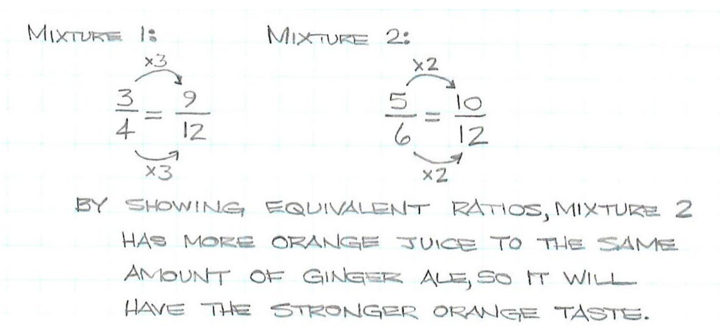

Figure 5. Possible student strategy

This response is similar to the previous strategy. While both solutions show using multiple copies of the initial ratio, this version is more abstract and represents using multiplicative reasoning to find a common quantity to do the comparison. As the coach, support teachers in thinking about how they respond to viewing this solution. Some teachers may say that they see what the student is thinking and do not need to ask questions. It is important that teachers learn what students are thinking and not assume. Initially, teachers should ask general questions regarding a student's solution. Next, if the student does not include it in their reasoning, teachers should ask specifics, such as why are they multiplying quantities in one ratio by the same value or why have they chosen the values of two and three with which to multiply? Here, students need to justify their actions with mathematical language including the importance of using the same number of sets to create equivalent ratios or using equivalent ratios to determine relationships that have common quantities. When possible, teachers can ask students to link their solution to a sample from another student (such as the solution strategy presented in figure 5). When students understand the learning goal with depth through high-cognitive-demand questions, they will be able to connect the visual representations of the ratios to the multiplying of quantities in their solution strategy.

Planning cannot always anticipate all of what teachers will see in a lesson. Teachers may need to ask questions of a student when they see an unexpected strategy to learn how the student is making sense of the problem and to determine if the strategy is mathematically correct. As a coach, it is helpful to support teachers by providing a strategy to discuss that was not considered during initial planning. For example, one unexpected strategy that a student might use is a version of the "butterfly method" (sometimes referred to as "cross multiplication strategy," see figure 6).

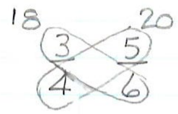

Figure 6. The "Butterfly Method"

While this strategy determines a relationship between the ratios (the number 18 – from multiplying 3 and 6 – is less than the product of 20 – determined by multiplying 4 and 5, so that the second ratio would have the stronger orange taste), this strategy may be employed by a student without understanding of what is represented. Here, the teacher should ask how the student knows that their work leads to making sense of the comparison between the two ratios. If the student replies with something about a rule that they have learned but not the mathematics that underlies the rule (Karp, Bush, & Dougherty, 2015), then the teacher can ask for another strategy or indicate that the student can only use strategies where they can justify the mathematics. This method is mathematically valid, so the teacher should ask the student about it. This strategy is actually a shortcut to determining a common quantity comparison. If the second quantities are multiplied, a common second quantity across ratios is created. Multiplying the first quantities by the second quantity of the other ratio to determine the 18 and the 20 ensures that the new ratios are equivalent to the original ratios. Now, equivalent ratios of $\frac{3}{4} = \frac{18}{24}$ and $\frac{5}{6} = \frac{20}{24}$ are created, so the "butterfly method" shows the different values of the first quantity for a common second quantity, allowing for the comparison of the two ratios. Again, students should be allowed to use shortcuts as long as they can justify the mathematics, and teachers need to be prepared to interpret both the reasoning and the validity of unanticipated strategies.

Students Developing Their Own Strategy

Throughout this discussion, the emphasis has been on asking students for their thinking and using their responses to guide teacher actions. This active engagement and listening are how teachers use evidence of student thinking to both guide their questioning and the lesson itself. Student responses support teachers to know what new questions to ask. They also help teachers to know when to "walk away" and allow the student to work on their own. In addition, collecting information from multiple students in the class helps teachers to determine when to pull the class together if many students need the same support and when to allow students to continue to share in small groups to help one another to progress on their individual strategies and solutions.

Extending Student Thinking

Now that we have talked about how to support students who need assistance in the development of their strategies, how can a teacher engage students who quickly generate a solution to the task and can share sound mathematical reasoning to support it? These students will need more. Accelerating is not always effective for these students, as moving faster can lead students to focus on answer-getting instead of sense-making. The key is for teachers to be prepared to support students to go deeper with the mathematics. How do coaches support teachers in this preparation?

There are several different possible strategies to use to enrich student thinking without acceleration. As a coach, consider how to encourage teachers to think about other ways students will be able to demonstrate understanding of the learning goal. Teachers could ask the student to find another way to determine the solution to the task. This can be done individually or by sharing ideas between students and encouraging them to make sense of each other's strategies. This tactic is useful with many tasks and should become part of the teacher's toolkit of extension prompts.

Another possible extension is for teachers to ask students to create a similar task in which students suggest using different quantities or different contexts, a different pair of ratios, and/or a different context. For the orange drink task, the coach might suggest that teachers have another pair of ratios available, perhaps comparing 7:12 with 7:13 or 8:11 with 9:11, and allowing students to both create their own context for a problem and use different strategies to compare ratios. In this way, students are deepening their knowledge and understanding of the learning goal.

Online Instruction

Finally, how does this scenario change if teachers are orchestrating an online classroom? As a coach, planning for online instruction should continue to focus on thinking about how teachers learn about student thinking in an online environment. Practice using the features of the online environment used by the teacher. Some techniques can be to use a "chat" feature of the video-sharing tool or collect information via a shared document. The important elements are that students are able to share their thinking in ways that are both comfortable and descriptive for the student. It is best when students can include drawings and words. One strategy could be to provide each student with the problem on their own Google slide or Google drawing and ask that they provide explanations of their work. They can even use video capture tools to provide their thinking as they are solving the task. Teachers have the opportunity to observe and monitor student work remotely and then either through breakout rooms or private chats ask questions about what they are doing.

Another element that needs to be considered when using remote instruction is to support teachers to plan how they want to implement questions. Teachers can place questions with the task from the start, they can decide to send questions to all students through the chat feature, or they can question students individually through discussion, notes on shared documents, or a private chat. Coaches support teachers to decide if they want students to have access to the work of other groups by creating one document in which all groups work or separate documents for each group. These options are similar to options available in face-to-face instruction and effective planning can support how these questions can be delivered. Each decision will have a different impact on the learning environment of the classroom. Of course, decisions made during the lesson may change how these questions are used, based on the evidence that is collected.

Wrapping It All Up and Next Steps

With the support of the coach, the teacher is ready to instruct the lesson. The learning goal has been clearly identified and linked to standards; the task has been solved, multiple solutions (both correct and incorrect) have been considered, and connections between strategies have been discussed; questioning has been evaluated; and anticipated student work has been linked to understanding of the learning goal. What would be best is for the coach to observe the lesson, perhaps even record the lesson, and then discuss with the teacher which predictions were accurate and what impact they had on the students' ability to achieve the learning goal. A recording of the lesson allows the teacher to evaluate the implementation of the lesson in realistic and meaningful ways in order to understand how well the planning session supports their teaching decisions. As a coach, you have the opportunity to discuss how the teacher poses purposeful questions, promotes student reasoning and problem solving, and solicits and uses student thinking throughout the learning process, all elements critical to creating effective student-centered learning environments.

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