## The Banneker Banner

## M C Maryland Council of Teachers of Mathematics

The Official Journal of the Maryland Council of Teachers of Mathematics

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The individuals below have given their time and expertise to read and review manuscripts submitted for this edition of the Banneker Banner. We are very grateful for their help.

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## Banneker Banner Submission Guidelines

The Banner welcomes submissions from all members of the mathematics education community, not just MCTM members. To submit an article, please attach a Microsoft Word document to an email addressed to strickland@hood.edu with "Banneker Banner Article Submission" in the subject line. Manuscripts should be original and may not be previously published or under review with other publications. However, published manuscripts may be submitted with written permission from the previous publisher. Manuscripts should be double-spaced, 12 point Times New Roman font, and a maximum of 8 pages. APA format should be used throughout the manuscript with references listed at the end. Figures, tables, and graphs should be embedded in the manuscript. As the Banner uses a blind review process, no author identification should appear on manuscripts. Please include a cover letter containing author(s) name(s) and contact information as well as a statement regarding the originality of the work and that the manuscript is not currently under review elsewhere (unless accompanied by permission from previous publisher). If electronic submission is not possible, please contact the editor to make other arrangements. You will receive confirmation of receipt of your article within a few days, and will hear about the status of your article as soon as possible. Articles are sent out to other mathematics educators for anonymous review, and this process often takes several months. If you have questions about the status of your article during this time, please feel free to contact the editor. Please note that photographs of students require signed releases to be published; if your article is accepted, a copy of the release will be sent to you and it will be your responsibility to get the appropriate signatures. If you would like a copy of this form at an earlier time, please contact the editor.

# Message from the President <br> Andrew Bleichfeld 

I am realizing that this will be the last Banner edition where I will be serving as President of this fine organization. Being the President of MCTM has been exhilarating, challenging, frustrating, exasperating, and a thousand other adverbs. But I have enjoyed it.

One of my colleagues told me that my legacy as President was the new Dine \& Discuss sessions being held around the state. I was not able to attend the session at Hood College, but I was able to attend in Howard County, at McDaniel College, and at Stoneleigh Elementary School in Baltimore County.

What I saw at all of these events was so many enthusiastic teachers giving of their own time to learn and discuss interesting topics. These events are intended to be quick, exciting, and rich professional development sessions. Although I witnessed three differently-managed events, all three were definitely quick and exciting and enriching.

It is so easy for educators to "poo poo" new trends in education, so easy for educators to hope that the education "pendulum" will swing the other way, so easy for educators to do the minimum necessary to get by. We all know those educators.

However, just the fact that you are reading this lets me know that you are not one of those educators. MCTM is looking to provide you and your colleagues with quality professional development. I believe that we are well on our way to doing that. More Dine \& Discuss sessions are being planned in Baltimore County, Baltimore City, Harford County, and Anne Arundel County, among others. Please look for those announcements and grab a colleague or two and attend! And, of course, planning for our Annual Conference, the mainstay of MCTM's professional development series, is already under way. Save the date of October $16^{\text {th }}$ at Baltimore Polytechnic Institute for this year's conference. It'll be A Walk in the PARCC. ©

While I still have the gavel in my hands, I would like to thank the great group of adults that serve on the MCTM Board. Give these folks an idea, they run with it. Give these folks a task, they deliver. And they are all volunteering their time. Talk about exhilarating! Watching these folks is indeed exhilarating. And include in that group the editor of the Banner, Tricia Strickland. She has put together some fine, informative issues for the members of MCTM. And please welcome Luis Lima as co-editor of the Banner to help infuse some more practical articles for future editions.

# Survey Results of Maryland Teachers who Instruct Elementary Students with Disabilities in Mathematics 

## By Tricia K. Strickland and Christy D. Graybeal Hood College, Frederick, MD

The contents of this article were developed under a grant from the US Department of Education, \#H323A120010. However, these contents do not necessarily represent the policy of the US Department of Education, and you should not assume endorsement by the Federal Government. Project Officer Tina Diamond.

Data from Maryland School Assessments in Mathematics demonstrate an achievement gap between students in special education and students in general education. As seen in Table 1, students in special education continually perform below students in general education and this gap increases as they progress through the middle school grades. This achievement gap is evident in high school as well. In 2014, 89.1\% of students in general education passed the Algebra/Data Analysis High School Assessment, while only 42.3\% of students in special education passed (Maryland State Department of Education [MSDE], 2014). Although

Maryland students are performing slightly better than students across the nation, data from the National Assessment of Educational Progress also indicate an achievement gap between students with disabilities and their non-disabled peers (MSDE, 2014; National Center for Education Statistics, 2013). See Table 2.

Table 1. Percent of students earning Proficient or Advanced scores on the Mathematics portion of the 2014 Maryland School Assessments

| Grade | Special Education | Regular Education | Gap |
| :---: | :---: | :---: | :---: |
| 3 | 39.9 | 78.1 | 38.2 |
| 4 | 45.6 | 85.0 | 39.4 |
| 5 | 33.2 | 78.0 | 44.8 |
| 6 | 26.8 | 73.0 | 46.2 |
| 7 | 22.4 | 68.2 | 45.8 |
| 8 | 17.1 | 63.7 | 46.6 |

Table 2. Percent of students earning Proficient or Advanced on the Mathematics portion of the 2013 National Assessment of Educational Progress

| Grade | National - <br> All <br> Students | Maryland - <br> All Students | National - <br> Special <br> Education | Maryland - <br> Special <br> Education | National <br> Gap | Maryland <br> Gap |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 41 | 47 | 18 | 19 | 23 | 28 |
| 8 | 34 | 37 | 8 | 10 | 26 | 27 |

Based on the data presented above, there is a critical need to improve the mathematics outcomes for all students, but especially for students with disabilities. The National Mathematics Advisory Panel (NMAP, 2008) reports that the mathematics achievement of students is directly related to the effectiveness of the teacher. Campbell et al. (2014) have identified a significant relationship between teachers' mathematical content and pedagogical knowledge and their students' mathematics achievement. Teachers must know the mathematics content they are teaching, as well as how to make connections to important mathematics prior to and after the level they are teaching. Additionally, teachers of students with disabilities must be knowledgeable of the evidence-based instructional practices regarding mathematics instruction so that they may implement these strategies with fidelity. Unfortunately, there is evidence of a long-standing research to practice gap in the field of special education (McLeskey \& Billingsley, 2008).

## Survey of Maryland Teachers

In the Spring of 2014, an exploratory study was undertaken to hypothesize areas in need of professional development regarding the teaching of mathematics to elementary students with disabilities (Strickland, 2015). The online survey was sent to 4,500 randomly selected elementary teachers in Maryland. Approximately 400 teachers responded.

Within the survey, teachers responded to a series of questions intended to assess their knowledge of research-based special education instructional practices. See Table 3 for a list of the practices, which were taken from Gersten and collegues’ (2009) meta-analysis of mathematics instructional interventions for students with learning disabilities. Additionally, a list of resources for each instructional practice is provided in Table 3.

An additional section of the survey required teachers to answer a number of questions intended to assess their pedagogical content knowledge in mathematics. These items focused on key elementary concepts, such as whole number operations, fractions, and decimals, and were aligned to the Common Core State Standards. See Table 4 for topics covered as well as resources. Table 5 provides sample questions for each section.

Results of Strickland's (2015) study suggested that Maryland teachers who provide mathematics instruction to elementary students with disabilities may not have a strong knowledge of special education research-based instructional practices, as evidenced by averaging only $41.5 \%$ accuracy on these survey items. Teachers scored only slightly better on the pedagogical content items, averaging 57.6\% accuracy. Additionally, Strickland disaggregated the results based on certification (i.e., Elementary Education, Special Education, and Dually Certified). Teachers certified in Elementary Education significantly outperformed teachers certified in Special Education and Dually Certified teachers on the items addressing pedagogical content knowledge. However, Elementary Education teachers averaged only 61.5\% accuracy on these items. There was no difference in performance on the special education items among the three groups, which is disturbing as Special Educators should be knowledgeable of researchbased instructional practices.

Table 3. Special Education Research-based Special Education Practices

| Research Supported Special Education Practice | Resources |
| :---: | :---: |
| Explicit Instruction | The Access Center's Direct or Explicit Instruction and Mathematics http://www.k8accesscenter.org/training_resources/DirectExplicitInstruction_Mathe matics.asp <br> Special Connections' Direct Instruction: Math <br> http://www.specialconnections.ku.edu/?q=instruction/direct_instruction/teacher_tool s/direct instruction math <br> IRIS Center, Vanderbilt University: High-Quality Mathematics Instruction: What Teachers Should Know: This module describes the components of high-quality mathematics instruction: a standards-based curriculum and evidence-based strategies. It also highlights several effective practices teachers can use to teach mathematics. Page 7 of the module focuses on Explicit Instruction. http://iris.peabody.vanderbilt.edu/math/chalcycle.htm <br> Hudson, P., \& Miller, S.P. (2006). Designing and implementing mathematics instruction for students with diverse learning needs. Boston: Pearson Education, Inc. |
| Enhanced Anchored Instruction | Teaching Enhanced Anchored Mathematics website: http://team.wceruw.org/index.html. <br> For additional suggestions and examples of using authentic contexts, please see the MathVIDS at http://www.coedu.usf.edu/main/departments/sped/mathvids/strategies/ac.html. |
| Schema-based Instruction | Jitendra, A., \& DiPippi, C. M. (2002). An Exploratory Study of Schema-Based Word-Problem-Solving Instruction for Middle School Students with Learning Disabilities: An Emphasis on Conceptual and Procedural Understanding. The Journal of Special Education, 36( 1), 23-38. Retrieved from http://www.ldonline.org/article/5678/. <br> Jitendra, A. (2007). Solving math word problems: Teaching students with learning disabilities using schema-based instruction. Austin, TX: PRO-ED. |
| Concreterepresentational/S emiconcrete Abstract Instruction | University of Kansas. Concrete-to-Representational-to-Abstract (C-R-A) <br> Instruction. Retrieved from <br> http://www.specialconnections.ku.edu/?q=instruction/mathematics/teacher tools/co <br> ncrete_to_representational_to_abstract_instruction. <br> Virginia Department of Education. Concrete-to-Representational-to-Abstract |


|  | sequence of instruction Retrieved from http://www.coedu.usf.edu/main/departments/sped/mathvids/strategies/cra.html <br> The Access Center. (2004). What Is the Concrete-Representational-Abstract (CRA) Instructional Approach? Washington, DC: American Institutes for Research. <br> Retrieved from <br> http://165.139.150.129/intervention/ConcreteRepresentationalAbstractInstructional Approach.pdf |
| :---: | :---: |
| Classwide Peer Tutoring | University of Kansas. Classwide peer tutoring. Retrieved from http://www.specialconnections.ku.edu/?q=instruction/classwide peer tutoring <br> IRIS Center, Vanderbilt University: High-Quality Mathematics Instruction: What Teachers Should Know: This module describes the components of high-quality mathematics instruction: a standards-based curriculum and evidence-based strategies. It also highlights several effective practices teachers can use to teach mathematics. Page 7 describes Peer Assisted Learning Strategies [PALS] Math, ClassWide Peer Tutoring for mathematics. <br> http://iris.peabody.vanderbilt.edu/math/chalcycle.htm |
| Peer Assisted Learning Strategy | Vanderbilt University. Peer Assisted Learning Strategies. Retrieved from http://kc.vanderbilt.edu/pals/ |
| Metacognitive Strategies | INQUIRE. Math Think Aloud. Retrieved from <br> http://nycdoeit.airws.org/pdf/Math\%20Think\%20Aloud.pdf <br> Math Vids. Metacognitive Strategies. Retrieved from http://fcit.usf.edu/mathvids/strategies/tms.html <br> A., Koedinger, K. R., \& Ogbuehi, P. (2012). Improving mathematical problem solving in grades 4 through 8: A practice guide (NCEE 2012-4055). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from http://ies.ed.gov/ncee/wwc/pdf/practice guides/mps pg 052212.pdf Pages 17-22. |
| Use of Heuristics / Strategy Instruction | INQUIRE (2005). LAP Fractions. West Virginia University Press. Retrieved from http://nycdoeit.airws.org/pdf/LAP\%20Fractions.pdf <br> The Access Center’s Mathematics Strategy Instruction (SI) for Middle School Students with Learning Disabilities by Paula Maccini and Joseph Gagnon http://digilib.gmu.edu/dspace/bitstream/1920/284/1/MathSIforMiddleSchoolStudent swithLD.2.pdf |
| Student verbalizations of their Mathematical Thinking | Garnett, K. (1998). Mathematics learning disabilities. Retrieved from http://www.ldonline.org/article/5896/. <br> Woodward, J., Beckmann, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, <br> A., Koedinger, K. R., \& Ogbuehi, P. (2012). Improving mathematical problem solving in grades 4 through 8: A practice guide (NCEE 2012-4055). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from http://ies.ed.gov/ncee/wwc/pdf/practice guides/mps_pg 052212.pdf Pages 10-16 |


| Progress <br> Monitoring | Curriculum Based Measurement Warehouse offers free internet based resources <br> http://www.interventioncentral.org/curriculum-based-measurement-reading-math- <br> $\underline{\text { assesment-tests }}$ |
| :--- | :--- |
| IRIS Center, Vanderbilt University: RTI Mathematics module contains information <br> regarding the use of Progress Monitoring in the Response to Intervention <br> framework. http://iris.peabody.vanderbilt.edu/rti_math/chalcycle.htm. |  |
| National Center on Progress Monitoring <br> conttp://www.studentprogress.org/default.asp <br> publications. |  |

Table 4. Mathematics Pedagogical Content

| Mathematics <br> Pedagogical <br> Content | Resources |
| :--- | :--- |
| Inappropriateness <br> of key word <br> strategy use | Video by Dr. Sybilla Beckmann on the Doing What Works website: <br> http://dwwlibrary.wested.org/media/word-problems |
| Relational <br> thinking | Molina, M., \& Ambrose, R.C. (2006). Fostering relational thinking while <br> negotiating the meaning of the equals sign. Teaching Children Mathematics, <br> 13(2), 111-117. Retrieved from : <br> http://sddial.k12.sd.us/esa/grants/sdcounts/sdcounts08- <br> $\underline{\text { o9/fall08/Equal_Sign.pdf }}$ |
| Forms of <br> equations; 9= <br> 5 | National Council of Teachers of Mathematics' Research Brief on "What do <br> students struggle with when first introduced to algebra symbols": <br> http://www.nctm.org/uploadedFiles/Research_News_and_Advocacy/Researc |
| $\underline{\underline{\text { h/Clips_and Briefs/research\%20brief\%2008\%20- }}}$\%20\%20initial\%20symbols.pdf |  |
| Contextualized <br> division problem | Common Core State Standard 4.OA.3 emphasizing the importance of <br> interpreting remainders: http://www.corestandards.org/Math/Content/4/OA |
| Order of <br> operations <br> (addition and <br> subtraction only) | National Council of Teachers of Mathematics activity on Order of <br> Operations: http://illuminations.nctm.org/LessonDetail.aspx?id=L730 |
| Associative <br> Property of <br> Multiplication | Common Core State Standards Table 3 on p. 90: <br> http://www.corestandards.org/assets/CCSSI_Math\%20Standards.pdf |
| Comparison <br> strategies for <br> decimals | Common Core State Standard 5.NBT.3.b: <br> http://www.corestandards.org/Math/Content/5/NBT |
| Use of various | Video by Dr. Jonathan Brendefur on the Doing What Works website: |


| models for <br> fractions | https://sites.google.com/site/emstlonline/dww/developing-effective-fractions- <br> instruction-for-k-8/multiple-interpretations-of-fractions |
| :--- | :--- |
| Contextualized <br> fraction <br> multiplication | IES Practice Guide on Developing Effective Fractions Instruction for <br> Kindergarten Through 8 8th Grade: <br> http://ies.ed.gov/ncee/wwc/practiceguide.aspx?sid=15 |
| Comparison <br> strategies for <br> fractions | IES Practice Guide on Developing Effective Fractions Instruction for <br> Kindergarten Through $8^{\text {th }}$ Grade: <br> http://ies.ed.gov/ncee/wwc/practiceguide.aspx?sid=15 |
| Contextualized <br> fraction division | IES Practice Guide on Developing Effective Fractions Instruction for <br> Kindergarten Through 8 $^{\text {th }}$ Grade: <br> http://ies.ed.gov/ncee/wwc/practiceguide.aspx?sid=15 |

Table 5. Sample Items

Special Education Evidence-Based Practices Assessment Items

Ms. Smith wanted to show her students how fun math can be, while also demonstrating the everyday uses of math. She split the class into groups and had them watch a video entitled, "Fraction of a Cost." The video shows three children who decide to build their own skateboard ramp. Ms. Smith's students needed to use their knowledge of measurement, fractions and money to respond to numerous problems linked to the video context. This is an example of which evidence-based practice?
A. Peer Assisted Learning
B. Concrete-Representational-Abstract graduated instructional sequence
C. Schema-based instruction
D. Enhanced Anchored Instruction
E. Progress Monitoring

Ms. Smith has decided to incorporate Classwide Peer Tutoring in her inclusive math classroom. She is going to place her highest achieving students with her lowest achieving students in hopes of improving the achievements of her struggling students. What is your response to Ms. Smith?
A. Tutoring pairs should consist of students who do not have an extreme difference in ability.
B. This is the best way to meet the needs of Ms. Smith's struggling students.
C. Peer tutoring is not an evidence-based practice so it should not be used.
D. Peer tutoring is always beneficial, regardless of how you pair students.

| Pedagogical Content Knowledge for Mathematics | Which of the following explanations is/are a valid way to compare $\frac{5}{7}$ and $\frac{9}{11}$ ? (check all that are valid) <br> A. Five is smaller than nine, so $\frac{5}{7}$ is smaller than $\frac{9}{11}$. <br> B. Sevenths are bigger than elevenths, so $\frac{5}{7}$ is bigger than <br> $\frac{9}{11}$. <br> C. $\frac{5}{7}=\frac{55}{77}$ and $\frac{9}{11}=\frac{63}{77}$, so $\frac{9}{11}$ is larger than $\frac{5}{7}$. <br> D. $\frac{5}{7}$ is two sevenths less than one. $\frac{9}{11}$ is two elevenths less than one. Since elevenths are smaller than sevenths, this means that $\frac{9}{11}$ is bigger than $\frac{5}{7}$. |
| :---: | :---: |
|  | Todd and Steven were simplifying the expression: $12-2+5$ Todd said, "The answer is fifteen because twelve minus two equals ten and then ten plus five equals fifteen." <br> Steven said, "No, the answer is five because you need to add before you subtract. Two plus five is seven and twelve minus seven is five." <br> Who is correct? <br> A. Todd is correct. <br> B. Steven is correct. <br> C. Neither is correct. |

## Implications

## Certification Requirements

Currently, the Maryland State Department of Education requires teachers certified in Elementary Education to earn 12 credits in mathematics (COMAR 13A.12.02.04). There is no specification about what areas of mathematics these 12 credits should be in. There is also no requirement that prospective elementary teachers take coursework on mathematics pedagogy. While the requirement of 12 credits in mathematics is aligned with recommendations made by the Conference Board of the Mathematical Sciences (2010) in the Mathematical Education of Teachers II (MET II), the non-specificity of these requirements is not aligned. MET II recommends:
"Programs designed to prepare elementary teachers should include 12 semester hours focused on a careful study of mathematics associated with the CCSS ( $\mathrm{K}-5$ and related aspects of 6-8 domains) from a teacher's perspective. ... It also includes some attention to methods of instruction. Number and operations, treated algebraically with attention to properties of operations, should occupy about 6 of those hours, with the remaining 6 hours devoted to additional ideas of algebra (e.g., expressions, equations, sequences, proportional relationships, and linear relationships), and to measurement and data, and to geometry. When possible, program designers should consider courses that blend the study of content and methods. Prospective teachers who have a limited mathematical background will need additional coursework in mathematics. It bears emphasizing that familiar mathematics courses such as college algebra, mathematical modeling, liberal arts mathematics, and even calculus or higher level courses are not an appropriate substitute for the study of mathematics for elementary teachers, although they might make reasonable additions. Also, it is unlikely that knowledge of elementary mathematics needed for teaching can be acquired through experience in other professions, even mathematically demanding ones" (pp. 31-32).

Considering the fact that only approximately 60\% of the Mathematics Pedagogical Content questions were answered correctly by participants certified in elementary education, we recommend that the Maryland State Department of Education consider revising certification requirements to ensure that future elementary school teachers have had mathematics coursework and pedagogical coursework relevant to the mathematics they will be teaching.

Similarly, Special Education teachers only answered approximately half of the Mathematics Pedagogical Content questions correctly. Currently Maryland certification in Special Education at the Elementary/Middle (grade 1 - grade 8) level does not include any requirements in mathematics content or mathematics pedagogy (COMAR 13A.12.02.20). Since many special educators are responsible for teaching mathematics and/or supporting mathematics instruction, this is worrisome. The Maryland State Department of Education may consider adopting mathematics content and pedagogy requirements as recommended by MET II. The MET II's recommendations for preparation apply to all teachers, including special education teachers (p. 37).

Note that the State of Maryland requires both Elementary Education and Special Education teachers to earn at least 12 credits in reading instruction. Maryland certification in Special Education for elementary / middle school requires candidates to complete (a) 3 semester hours in processes and acquisition of reading skills; (b) 3 semester hours in best practices in reading instruction that include the cuing systems of graphophonics, semantics, and syntactics; (c) 3 semester hours in the use of reading assessment data to improve instruction; and (d) 3 semester hours in materials for teaching reading in order to gain literary experience, to perform a task, and to read for information (COMAR 13A.12.02.20). However, COMAR does not explicitly require any coursework specific to mathematics instruction. Since mathematics and language arts are the foci of the elementary grades, it is essential that all teachers are knowledgeable of the content they will be teaching as well as best practices. Thus, in addition to specifying requirements in reading instruction, the State should look into specifying requirements in mathematics instruction.

Based on the results of Strickland's (2015) survey, teachers of students with disabilities may not have a strong knowledge of research-based special education mathematics practices. Special education teacher education programs in Maryland Institutions of Higher Education vary regarding mathematics pedagogy coursework. University of Maryland, College Park requires all special education teacher candidates to complete a course entitled EDSP485/683: Assessment and Instruction in Mathematics in Special Education. This course is specifically designed to teach candidates to use Special Education research-based practices in mathematics. In contrast, Towson University requires candidates to complete a course entitled MATH 323: Teaching Math in Elementary School. This course appears to be a general education mathematics course, based
on a review of a syllabus posted online (http://pages.towson.edu/shirley/math323.htm). There is no mention of special education research-based practices within this syllabus.

We recommend that the Maryland State Department of Education consider revising certification requirements to ensure that future special education teachers have mathematics pedagogy coursework specifically addressing special education mathematics research-based practices.

## Professional Development

While changes in certification requirements may help to better prepare future teachers for the demands of mathematics instruction, current teachers also need support. Several teachers in Strickland's (2015) study wrote comments mentioning the desire for professional development on special education practices specific to mathematics instruction. For example, one teacher stated, "I think professional development on successful instructional strategies for inclusive classrooms and on co-teaching models (beyond the one teach, one support model) would be beneficial. Mathematics is definitely an area I think all educators (not just special educators) need additional professional development in with strategy instruction." and another stated, "I would love better PD's to teach math in new ways which PARCC will require."

## MET II recommends:

"Once they begin teaching, elementary teachers need continuing opportunities to deepen and strengthen their mathematical knowledge for teaching, particularly as they engage with students and develop better understanding of their thinking. Professional development may take a variety of forms....the best professional development is ongoing, directly relevant to the work of teaching mathematics, and focused on mathematical ideas..."(CBMS, 2010, p. 32).

The MET II's recommendations for professional development apply to all teachers, including special education teachers (p. 37). Thus, we recommend that ways to continue to
support the development of teachers' pedagogical knowledge, content knowledge, and pedagogical content knowledge be explored.

Additionally, professional development is needed regarding special education evidencebased instructional practices, specifically relating to Schema-based Instruction, Enhanced Anchor Instruction, Explicit Instruction, Concrete-Representation-Abstract instruction, and Peer Assisted Learning Strategy. Although researchers in Special Education have made substantial progress identifying evidence-based practices, there is a long-standing research-to-practice gap in special education. Cook, Cook, and Landrum (2013) identify the non-effective methods of disseminating research as the main reason for this gap, stating that establishing evidence-based practices "is likely to have limited impact on teaching practices and student outcomes" (p. 164) due to passive dissemination of research. It is recommended that institutes of higher education work cooperatively with their local school systems to provide professional development in special education research-based instructional practices to all elementary teachers instructing students with disabilities in mathematics.

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# Connecting Research to Practice: Pizza Party Planning Project 

By Kathryn Walsh

Montgomery County Public Schools


Our pizza party is approaching quickly. We want to plan everything down to the last detail, including how many pizzas to buy. All of the pizzas are the same size, but they are all cut differently. We can get pizzas that are partitioned into fourths, sixths, or eighths. If we buy the pizzas that are partitioned into fourths, you will each get $1 / 4$ of a pizza. If we buy the pizzas that are partitioned into sixths, you will get $2 / 6$ of a pizza. If we buy the pizzas that are partitioned into eighths, you will each get $3 / 8$ of a pizza.

We are making the following assumptions:
$\rightarrow$ Everyone in class can and will eat the same amount of pizza.
$\rightarrow$ Everyone in the class LOVES pizza and wants to eat as much as possible.
$\rightarrow$ Everyone is present in class the day of the party.


Work with your partners to determine which pizzas we want to order. Then, work with your partners to figure out how many pizzas we need to order to feed all 26 students plus Mrs. Walsh. Be able to explain your thinking to the other groups using some sort of model and math vocabulary.

See p. 41 for an explanation of the solution and examples of student engagement.

# Teaching the Function Concept in a Technology-Rich \& Common Core-aligned Classroom <br> By Thomas Coleman and Luis Lima Baltimore City Public Schools 

The Common Core State Standards for Mathematics (CCSS-M) embodies a divergent vision of mathematics teaching and learning than is typically enacted in K-12 schools throughout the United States. This set of standards calls for increased rigor, focus, and coherence in mathematics instruction while advocating against the prevalent "mile wide and inch deep" approach currently found in American classrooms (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGACBP], 2012). Certainly, the CCSS-M's call for reform in the teaching and learning of mathematics is not entirely new (National Council of Teachers of Mathematics, 2000). The CCSS-M is novel due to the overwhelming adoption of these standards within the United States' K-12 educational system and due to the set of assessments designed as accountability measures toward determining students’ college and career readiness (CCSSO, 2013).

Paramount in this shift is the CCSS-M's explicit identification of eight Standards of Mathematical Practice, or SMPs, that "describe varieties of expertise that mathematics educators at all levels should seek to develop in their students" (NGACBP, 2012, p. 6). These SMPs outline competencies students should demonstrate related to content and procedural mastery in ways they may have never been required to before. This, in turn, requires teachers to shift their pedagogy in order to foster such practices.

Perhaps the place where these shifts are most apparent is in middle school mathematics. Specifically, the CCSS-M charges middle school students to learn and deeply understand concepts pertaining to equations and functions (Conference Board of Mathematical Science, 2012) that they were not required to attain previously. For example, in $6^{\text {th }}$ grade the CCSS-M states that instructional time should be spent on the
critical area [... of] writing, interpreting, and using expressions and equations [... so that] students know that the solutions of an equation are the values of the variables that make the equation true. Students use properties of operations and the idea of maintaining the equality of both sides of an equation to solve simple one-step equations. (NGACBP, 2012, p. 39)

This and other related understandings are continually developed throughout the $7^{\text {th }}$ and $8^{\text {th }}$ grades resulting in students being expected to "use linear equations and systems of linear equations to represent, analyze, and solve a variety of problems" (p. 52). Additionally, in the $8^{\text {th }}$ grade students are charged with attaining the concept of function and how functions can be used to describe quantitative relationships. Such understandings, and their required depth, were previously the main focus of high school algebra courses.

Thus, the focus of this article is to provide teachers with a sample activity that highlights opportunities to deepen and strengthen their own and their students’ mathematical knowledge of the CCSS-M Equations and Functions domains present in grades 6 through 8. These domains were selected due to their central importance in the study of mathematics (Dubinsky \& Harrel, 1992; Eisenberg, 1992; Elia, Panaoura, Gagatsis, Gravvani, \& Spyrou, 2008), mathematical complexity (Chazan \& Yerushalmy, 2003), and emphasis given to them within the CCSS-M. Some have gone as far as to assert, "function is the only pedagogically necessary and desirable object in [...] the school subjects of algebra, trigonometry, probability and statistics, pre-calculus and calculus" (Schwartz \& Yerushalmy, 1992, p. 1).

## Multiple Representations in the Learning of Functions

Representational fluency is a vital component in understanding the content described by the CCSS-M Equations and Functions domains. Standard 6.EE.9, for example, charges students to "analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation" (NGACBP, 2012, p. 44, italics added). Consistent with its explicit calling out of representational fluency, the CCSS-M states, "mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs" (p. 6). Indeed, the ability to
flexibly and accurately move between mathematical representations is at the core of not only functional understanding (NCTM, 2000), but also mathematical understanding (Arcavi, 2003).

Representational fluency is important because each representation of functions (algebraic, graphical, and tabular) highlights different characteristics. Graphs, for example, often show the global characteristics of a function, such as rate of change and asymptotes, but are deficient in accuracy and can be influenced by other factors, such as scale. Tables act as a quick and precise reference for a selection of individual points, but often result in an insufficient view of a function's global nature due to a lack of generality. Equations are useful in finding specific values not apparent in either the graph or table, such as $f(\pi)$ or $f(x)$ for a specific $x$, but may "blur or obstruct the mathematical meaning or nature of the represented objects and cause difficulties in some students' interpretation of their results" (Friedlander \& Tabach, 2001, p. 174). To clarify this idea, consider a student presented with a tabular and graphical representation of the same continuous function. From the table, the student may conceptualize the function as a group of discrete points that satisfy the functional relationship. Alternately, the same function may be interpreted as a continuous object, with no points at all, when viewed in the graphical representation. Each conception of the function is markedly different and leads to a limited view of the given function.

Thus, equations, graphs, and tables each have great value in depicting the same functional relationship since "the content of a representation depends more on the register of the representation than on the object represented. That is the reason why passing from one register to another changes not only the means of treatment, but also the properties that can be made explicit" (Duval, 2006, p. 114). This helps explain why students may have difficulty recognizing that the multiple representations of a function refer to the same concept and do not constitute different or autonomous mathematical objects (Arnold, 2005; Elia \& Spyrou, 2006). Students’ understandings of functions may become compartmentalized by different representations and disassociated from a common concept. Nearly by definition, this compartmentalization lessens the connections that students make between the different characteristics of
functions that each representation affords. In fact, many empirical studies support the conclusion that students have compartmentalized understandings of function (Moschkovich, Schoenfeld, \& Arcavi, 1993; Elia \& Spyrou, 2006) which results in students experiencing difficulty in translating between the algebraic, tabular, and graphical representations of functions (Carpenter, Corbit, Kepner, Lindquist, \& Reys, 1981; Leinhardt, Zaslavsky, \& Stein, 1990; Markovits, Eylon, \& Bruckheimer, 1986).

Students’ difficulty in translating between representations has significant implications for their understandings of the concept of function. As discussed previously, each representation illuminates certain characteristics of the function they depict (NCTM, 2000) and has inherent disadvantages in their depictions (Friedlander \& Tabach, 2001). It logically follows then that if students are able to utilize multiple representations in an integrated manner they will be able to gain a more thorough picture of the function concept (Elia et al., 2008) as the combined use of all representations "can cancel out the disadvantages and prove to be an effective tool" (Friedlander \& Tabach, 2001, p. 174) to understand function. This is in contrast to relying on only one representation, which results in a limited view of the conception of a function. Traditionally, secondary mathematics instructors have primarily focused their instruction on the use of algebraic representations of functions rather than the graphical representation (Elia \& Spyrou, 2006, p. 257). Hence, students' difficulty in the construction of the function concept is linked to the limited representations utilized when learning it.

## Pedagogical implications.

Moving toward a non-compartmentalized notion of function is dependent upon broadening the scope of meaningful interactions that students have with the varying representations and aspects of functions. The preeminence of the algebraic representation of functions in traditional mathematics instruction focuses on simply translating the algebraic representation into the graphical representation and not on the affordances of the latter apart from the former. As a consequence, students often view graphs as the product of a process they must complete and not as a representation that can be utilized for further discovery, analysis, and interpretation (Elia et al., 2008).

One can assume a different instructional approach, however. Teaching for connections and understanding speaks to the heart of the Common Core's dual-intensity shift toward conceptual and procedural proficiency, as well as the to behaviors of the mathematically competent person codified by the SMPs. Thus, in the context of developing the concept of functions, it is not enough to teach how to procedurally convert from one representation to another. Instead,
[w]e may pursue a different pedagogic approach to the learning and teaching of algebra as a formal mathematical system. Rather than starting, as we normally do, with particular functions expressed symbolically, and then, almost as an afterthought, turning to the graphs of those functions, we turn from the outset to particular functions expressed both symbolically and graphically. Because we have independent ways of manipulating both the symbolic representation and the graphical representation of a function... it becomes feasible to ask questions about the invariant properties of those families. (Schwartz \& Yerushalmy, 1992, p. 288) In this pedagogical approach both the symbolic (equations) and graphical representations are seen as being a source of learning, not one being secondary to the other. Through the exploration of both representations students will be able to learn about the common characteristics of functions that each representation displays in their own unique way. This approach should explicitly draw out connections between these representations, lessening or eliminating students' compartmentalized understanding of function. "We should teach for connections and understanding, not merely for procedural skills" (Moschkovich et al, 1993, p. 98).

This is not meant to suggest that the algebraic representation be abandoned altogether. Instead, it should be used concurrently with the other functional representations in order to allow studens to gain a more robust and complete notion of the function concept. Consequently, it is important for pedagogy to emphasize the affordances of and connections between each functional representation while not favoring any one representation in particular. Additionally, the CCSS-M makes explicit recommendations toward the integration of representational fluency throughout the middle and high school standards. In grade 8,
for example, standard 8.F.B. 4 calls for the construction of a function "to model linear relationships between two quantities [...] from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph" (NGACBP, 2012, italics added). Similarly, in Algebra I, there is a cluster dedicated to the analysis of "functions using different representations" (NGACBP, 2012, italics added).

Clearly, in order for teachers to facilitate such rigorous learning of the function concept they must possess the understandings toward which they are attempting to lead students. Teachers must be knowledgeable about each of these representations and the connections between them if they are to teach students in the manner described by the CCSS-M. The activity we propose in this article is meant to model a possible means of facilitating functional learning in a rigorous manner that leverages the multiple representations in which functions occur.

## Dynamic and Interactive Software in Mathematics Learning

The activity featured in this article was developed using the GeoGebra ${ }^{\text {TM }}$ software technology. We selected this specific technology due to its widespread accessibility, affordability, and ability to display multiple, dynamically linked, representations of the same function simultaneously. The use of technology to interactively link the multiple representations of functions in this way has been suggested to increase functional understandings (Havelková, 2013; Pierce, Stacey, Wander, \& Ball, 2011; Swartz \& Yerushalmy, 1992) and is recommended as the preferred means of engaging students in learning about functions (Schwartz \& Yerushalmy, 1992).

Jenkins (2007) suggests that instant feedback, such as that which is enabled by the interactive and dynamic linking of representations, encourages "play" in the problem-solving process. Play in this context has been found to be beneficial since "part of what makes play valuable as a mode of problemsolving and learning is that it lowers the emotional stakes of failing: players are encouraged [...] to take risks and learn through trial and error" (p. 100). Using trial and error in this way encourages students to explore non-canonical solution paths in a low risk technological environment. Hahkioneimi and Leppaaho
(2012) also emphasize that, "this software makes it possible to try different kinds of solution methods which would be too inconvenient with paper and pencil. Thus, GeoGebra encourages students to try out multiple ideas, as well as to make conjectures and to test them" (p. 26).

This technology's computational and computer algebra system (CAS) functionalities also allow students to efficiently and accurately progress through activities without having to complete a number of more procedural skills, such as algebraic manipulations, producing graphs, or populating tables. GeoGebra's ability to perform these aspects of students' explorations is an important aspect of distributed cognition. "Work in distributed cognition focuses on forms of reasoning which would not be possible without the presence of artifacts or information appliances [such as technologies], which expand and augment human's cognitive capacities" (Jenkins, 2007, p. 106). In other words, participants may not be able to cognitively focus on or process the conceptual understandings of the learning activities they are intended to grapple with if their attention was focused on performing the more procedural aspects of those activities. Hence, GeoGebra's instantaneous production of graphs and tables frees the user from having to complete cognitively draining tasks. CAS also frees participants' cognitive load of the work required by symbolic manipulations. The use of GeoGebra in this activity is intended to expand and augment the learners' cognitive capacity, allowing them to focus on learning about equation and function concepts in a deep and rigorous way.

## Description of the Exploration Activity

The design and implementation of the activity was informed by the five practices for facilitating inquiry mathematics discussions. Here, the instructor is charged with
(1) anticipating likely student responses to cognitively demanding mathematical tasks, (2) monitoring students' responses to the tasks during the explore phase, (3) selecting particular students to present their mathematical responses during the discuss-and-summarize phase, (4) purposefully sequencing the student responses that will be displayed, and (5) helping the class
make mathematical connections between different students' responses and between students’ responses and the key ideas. (Stein, Engle, Smith \& Hughes., 2008, p. 321)

Overall, the teacher acts as a facilitator of the learning process, helping increase students’ autonomy, competence, and relatedness and thus increasing students’ intrinsic motivation (Jones, Uribe-Florez, \& Wilkins, 2011). In enacting each of these practices the facilitator aims to elicit, draw attention to, and facilitate discussions around the learners' wonderings and findings that may be potentially beneficial for the groups' learning. Following these practices is particularly important since the use of technology similar to GeoGebra is suggested to increase the complexity of mathematical discourse (Manouchehri, 2004).

## Conceptually exploring various methods of solving equations.

The activity will center on students being challenged to think of solving equations in a potentially novel way. "One can think of the solutions to an equation as the initially unknown values in a...domain for which two functions have equal outputs" (Chazan, 1993, p. 22). Thus, the equation (3/4) $x-1=(7 / 5)$ can be thought of as a question asking for what values in the domains of the functions $f(x)=(3 / 4) x-1$ and $g(x)=(7 / 5)$ produce the same output.

Conceptualizing the equation as an equivalence of two functions is an aspect in appreciating the complexity of this content material (Chazan \& Yerushalmy, 2003) and is consistent with CCSSM standard A-REI-D. 11 .

Using the GeoGebra software provides a bounded and semi-structured environment where students can explore and learn using this conception of equations. The interface used for this activity includes a screen vertically divided into three windows (see figure 1). The left of the interface displays the algebra window, the graphics (graph) window is in the center, and the right of the interface displays the spreadsheet (table) window. Within the algebra window there will be two functions, namely $f 1(\mathrm{x})$ and $g 1(\mathrm{x})$. Here, the function $f 1(x)=(3 / 4) x-1$ represents the left hand side, with $g 1(x)=(7 / 5)$ the right side,
of the original equation $(3 / 4) x-1=(7 / 5)$. (The " 1 " present on $f 1$ and $g 1$ is meant to indicate that these are the functions depicting the first step of solving the equation $(3 / 4) x-1=(7 / 5)$. Thus, $f 2$ and $g 2$ would indicate the second step, etc. . .). Inputting these functions results in the graphs shown in the graphics window. We have also pre-programmed the spreadsheet window to be populated with a selection of points from these two functions. Each representation of these functions is colored red to indicate that they represent the first step of solving. The second step will use a different color. Lastly, the intersection of these two graphs is displayed in the graph and the algebra window point $A(3.2,1.4)$ in figure 1 . The functions found in the algebra window will be manipulated and the resulting new functions will be inputted. The results of any manipulations or additions will automatically be reflected in the graphic window.


Figure 1. GeoGebra Interface

GeoGebra will then be used to demonstrate how using the canonical solution method for solving this equation affects the graphs of the functions $f(x)$ and $g(x)$. The teacher will demonstrate how to first add 1 to both $f 1(x)$ and $g 1(x)$ using the interface, resulting in $f 2(x)$ and $g 2(x)$. In figure $2 f 2(x)$ is already entered, while $g 2(x)$ is being entered in the input bar at the bottom of the interface. Figure 3 shows the
entire completed step. This will be followed by dividing $f 2(x)$ and $g 2(x)$ by (3/4), resulting in $f 3(x)$ and $g 3(x)$ (figure 4). By displaying the intersection point of $f 3(x)$ and $g 3(x)$ we confirm that $x=3.2$.


Figure 2. Adding 1 to $f 1(x)$ and $g 1(x)$


Figure 3. Result of Adding 1 to $f 1(x)$ and $g 1(x)$


Figure 4. Result of Dividing $f 2(x)$ and $g 2(x)$ by (3/4)
The teacher will then ask if this is the only mathematically valid solution path. Hypothesized alternative methods will be elicited from students and recorded in a publically visible location. The teacher will not, however, comment on the validity of any of the participants' solutions. Next, students will explore the posited solution paths, and any others that they create, in their small groups using the GeoGebra file. The teacher will ask students to take note of their findings, confusions, conjectures, questions, and insights as they explore how to solve this equation. As the teacher circulates around the room he or she will also probe students, as needed, with questions pertaining to the idea of a solution set, how their explorations are affecting the two functions, and the connection between the multiple representations they are exploring. The teacher will also take note of students' explorations so that the group discussion can be purposefully planned and executed (Stein et al., 2008).

After the small group exploration, the whole group will discuss their findings, confusions, conjectures, questions, and insights surrounding their explorations. The teacher will direct this conversation according to the practices for facilitating inquiry mathematics discussions (Stein et al., 2008). This may entail restating participants’ thoughts, posing
questions, and focusing the direction of the conversation (see Cohen, 2004 for example). A focus of the whole group conversation will be on how students engaged in this transformational algebraic activity (Kieran, 2007) and what they learned as they "develop meaning for equivalence [...and] the use of properties and axiom in the manipulative processes themselves" (p. 714). It is not expected that students will fully develop this knowledge immediately, as cognitive research suggests they are quite challenging to obtain (Chazan \& Yerushalmy, 2003). It is anticipated, however, that the whole group discussion would provide more questions, conjectures, and methods of solving equations for later exploration within the GeoGebra environment. Further instruction and exploration is likely to be needed to solidify students' understandings.

## Possible Student Explorations

We anticipate (Stein et al., 2008) that certain explorations are likely. These explorations are based on methods the students are likely to have attempted. The hypothesized explorations are also grounded in areas where teachers may have curiosities. Below we explain a selection of these explorations and the possible understandings students may construct as a result. These descriptions are cursory and not meant to be exhaustive. Instead, by presenting these examples we hope to portray possible non-canonical steps students may consider in their explorations.

## Multiply by the Least Common Denominator

A common strategy used when solving equations that contain fractions is to multiply the entire equation by the least common denominator (LCD). To solve this particular equation, this strategy can be used after completing the canonical first step of adding 1 to both sides (see figures 5 and 6 ) or as one's first step (see figure 7).


Figure 5. Adding one to both sides as the first step


Figure 6. Multiplying by the LCD as the second step


Figure 7. Multiplying by the LCD as the first step
The purpose of multiplying by the LCD is often to eliminate all fractions found in the equation. Students, and teachers for that matter, often try to avoid operations with fractions. Thus, this strategy is a popular deviation from the canonical method of strictly using inverse operations to isolate the variable.

## Multiply by a Denominator not the Least Common Denominator

A similar strategy can be used to eliminate some, but not all, of the fractions in an equation.
Here, one multiplies the entire equation by one of the denominators within the equation that is not the
LCD. For our equation, one may multiply by either 4 (see figure 8 ) or 5 .


Figure 8. Multiplying by a denominator other than the LCD

This strategy eliminates the fractions for which the multiplied number is a multiple of that fraction's denominator. This method may be helpful in selectively eliminating fractions. Alternatively, this method may result from attempting to multiply the equation by the LCD but unintentionally neglecting one denominator when calculating it.

## Common Student "Mistakes"

There are certain common "mistakes" that students make when solving equations for a given variable. We should emphasize here that these manipulations are often viewed as mistakes, but they may be mathematically valid. All too often, however, students and teachers alike execute these manipulations incorrectly, which leads people to believe that the step itself is incorrect. This, however, is not the case. One of these "mistakes" occurs when students perform an inverse operation on a number that is on the opposite side of the equation as the variable. In this case, a student may subtract (7/5) from both sides of the equation (see figure 9). Another common "mistake" is when a student performs the same operation that is already present on both sides of the equation. For this equation, a student may subtract 1 from both sides (see figure 10) or multiply by (3/4). These steps and others like them may not assist in isolating the variable $x$. Thus, they are often viewed as mistakes because performing them may not contribute to a solution procedure that is as efficient as possible. Done correctly, however, these steps maintain the equality found in the equation. Thus, they are not actually incorrect manipulations.


Figure 9. Using an Inverse Operation on the "Wrong" Side of Equation


Figure 10. Performing the Same Operation instead of the Inverse Operation

## Learning Outcomes

In this section we describe the patterns we anticipant students will notice, as well as the potential understandings they may construct, during this learning experience.

## Graphical Effects of Algebraic Manipulations

Many of the manipulations one performs on the equation result in the graphs of the left and right hand sides of that equation changing. For example, when a student added 1 to both sides as a first step, the graphs of both the left and right hand sides shift vertically one positive unit (see figures 2 and 3 ). Then, when a student multiplied the equation by the LCD as a second step, the slope of the graph for the left hand side function increased and graph of the right hand side function moved out of the handheld's window (see figures 5 and 6). It seems that manipulating algebraic representation also affects the graphical representation.

This is not always the case, however. Consider, for example, the situation where a student first multiplies the equation by the LCD. The algebra window produces functions that are in factored form (see figure 11). Each side can be simplified using the distributive property and re-entered (see figure 12). This results in overlapping graphs for $f 2(x)$ and $f 3(x)$, as well as $g 2(x)$ and $g 3(x)$. Using the distributive property here does not change the graphs of the left and right hand side functions.


Figure 11. Multiply by LCD


Figure 12. Simplifying Using the Distributive Property
Therefore, the graphs of the left and right hand side functions may change when the equation is manipulated, but they don't have to. This is because certain types of manipulations do different things to the equation and its component functions. When one performs an operation on an entire equation (i.e. adding, subtracting, multiplying, dividing, squaring, square rooting, etc) he or she is transforming the functions representing each side of the equation. Thus, when one graphs these functions, which we are representing as the $f(\mathrm{x})$ and $g(\mathrm{x})$ within this GeoGebra file, the graphs are transformed. When one manipulates only one side of the equation using mathematical properties (i.e. distributive, associative, commutative, etc), however, he or she is not fundamentally changing the corresponding function. Thus, the graphs are unaffected.

## Graphical Intersection Point

In solving the equation one may notice that the $y$ coordinate of the intersection point may change, but the $x$ coordinate always remains the same. Further, the $x$ coordinate represents the solution to the equation. These observations hold true for all mathematically valid manipulations of the equation. First, to understand why the $y$ coordinate may move, consider the simple equation $7 x=42$. To solve this equation, one would divide both sides by 7 resulting in $x=6$. Let us consider how this step affects the
graphs of the left and right hand side functions. The original graphed functions would depict $f 1(x)=7 x$ for the left hand side and $g 2(x)=42$ for the right hand side of the equation. After dividing both sides by 7 we have $f 2(x)=x$ for the left hand side and $g 2(x)=6$ for the right hand side. Clearly the graphs of $f 1(x)=$ $7 x$ and $f 2(x)=x$ are different. This is also true for the graphs of $g 1(x)=42$ and $g 2(x)=6$. A similar argument can be made any time an operation (i.e. adding, multiplying, squaring, etc) is used on an entire equation.

Now, consider why the $x$ coordinate does not change. Again consider the equation $7 x=42$. We know that there is only one real number solution for $x$ since this is a first-degree polynomial. When one uses any mathematically valid operation to manipulate this equation an equivalent equation is produced and the one real number solution for $x$ is preserved. This includes the equation where the variable $x$ is isolated and the solution can easily be read. One may choose to increase the degree of this equation by multiplying by a power of $x$. This operation may introduce extraneous solutions. Regardless, this operation preserves the original solution for $x$ in the equation.

## Using the Order of Operations is NOT Required

When using the common method to solving equations one uses the inverse order of operations. Thus, one addresses addition and subtraction first, multiplication and division second, exponents third, and finally parenthesis and other grouping symbols. Following this order is not required, however. One may perform operations in any order as long as the operation is done correctly, thus maintaining the equality underlying the original equation. For example, to solve our equation one may divide the entire equation by (3/4) as a first step, before addressing the subtraction that is present (see figure 13). Important here, though, is that this division is done correctly; the entire left hand side of the equation must be divided by (3/4).


Figure 13. Performing Division First
This is a perfectly valid mathematical step and, as long as other valid steps are taken, will lead to the desired solution.

## Conclusion

We have argued for and justified the importance of the teaching and learning of equations and function to meaningfully include multiple representations (i.e.: tables, graphs and equations). Doing so increases students' understandings of the function concept in a manner that is pedagogically consistent with NCTM's principals and standards, CCSS-M's SMPs, and mathematics education research findings and recommendations. We then described an activity embodying such pedagogy. Completing this activity within the GeoGebra interface is consistent with recommendations for the transformation (i.e.- modifying symbolic and graphical representations) and comparison of functions to occur in technological environments where the symbolic and graphical representations are dynamically linked. This allows students "to understand more readily what is meant by a "solution set' and to be able to view equations, identities and inequalities in a coherent fashion all as examples of different kinds of comparisons
of functions" (Schwartz and Yerushalmy, 1992, p. 4). We sincerely hope that this discussion has provided insight into the Equations and Functions domains outlined in the CCSS-M and thoughts on ways to actively involve students in making sense of this content.

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## Pizza Party Project - Explanation By Kathryn Walsh

Based upon the information in the problem, we want to order the pizzas that are partitioned into eighths. Every student will get $3 / 8$ of a pizza. Working with groups, the students needed to prove that $3 / 8$ was the largest portion available. Different student groups chose different methods of comparing the fractions. Students chose Cuisenaire rods, Fraction Factory pieces, extended pattern blocks, and linking cubes to prove this.

For the second part of the problem, students needed to determine how many pizzas to order. There are twenty-six students plus the teacher eating, so there needs to be enough pizza for 27 people to eat. That requires $81 / 8$ or 81 slices of pizza for each person to get $3 / 8$ of a pizza. Students chose linking cubes, drawings, multiplication and division, and tallies to prove this.

To get 81 slices of pizza, we need to order 11 pizzas. The first 10 pizzas, partitioned into eighths, gives us 80 slices. However, that last slice requires us to order one additional pizza, so the answer is 11 . Students that used linking cubes made groups of eight, while students that made tallies tried to circle the groups. The students that made drawings counted the pizzas.

This project aligns to the Maryland College and Career Readiness Standards 4.NF. 2 and 4.NF.3.d

# Multiple Methods of Solving Quadratic Equations: An Algebra 

 One Project Idea Consistent with the Common CoreBy Doreen Hamm
Montgomery County Public Schools

Research reported in this paper was completed in a capstone project for the master's degree in mathematics
education in the Graduate School of Hood College.

This past year, I implemented the Draft Maryland Common Core State Curriculum
Framework for Algebra One High School at my high school. I also completed my capstone paper towards a Masters in Mathematics Education. I decided to tie the two together: For my capstone, I chose a topic that would give me a better background for future implementation of the Common Core State Standards into my classroom, Multiple Methods of Solving Quadratic Equations: Algebra One and the Common Core. As part of the paper, I created a student project to implement into my Algebra One classroom that I felt had captured the philosophy of the Common Core State Standards of Knowledge and the Common Core Standards for Mathematical Practice. I wanted to develop something that would be useful for this year and subsequent years for the Algebra One quadratics unit.
"The Standards of Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important 'processes and proficiencies' with longstanding importance in mathematics education."
(National Governors Association. 2010) In particular this project incorporates the following Standards of Practice:

CCSS.MATH.PRACTICE.MP5: Use appropriate tools strategically. Students are asked to examine the solution of a quadratic equation both by algebraic manipulation with paper and pencil and graphically using technology and representing the solution on graph paper.

CCSS.MATH.PRACTICE.MP6: Attend to precision. Students are asked to explain multiple solutions in their own words as they show step by step how to arrive at the same answer using multiple methods.

CCSS.MATH.PRACTICE.MP7: Look for and make use of structure. Students examine how mathematically they are constructing arguments using different algebraic reasoning and developing the experience to select the best method for each problem. (CCSS, 2010)

The simple procedure of solving a quadratic equation multiple ways was an eye opener for me. I was impressed with the ability of my students to explain their thinking step by step that was built into this project. On the other hand, I was also surprised that several students had to be prompted that if they got three different answers to the same equation they had to go back and find their errors as the same equation should have the same solution no matter how you choose to solve it. I believe true learning took place as a result of each student's investigation into multiple methods. I also encourage the use of technology to verify solutions. For some students the graphing calculator was their check, for others equation solvers available through the internet. Through teaching with the use of multiple methods, students are taking responsibility for their own learning. And I also realized that this project addresses the first Standard for Mathematical practice, CCSS.MATH.PRACTICE.MP1:

## CCSS.MATH.PRACTICE.MP1: Make sense of problems and persevere in problem

solving. In particular, this practice tells us that, "Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, 'Does this
make sense?’ They can understand the approaches of others to solving complex problems and identify correspondences between the different approaches."

I believe this project also captures the essence of the Universal Design for Learning (NCUDL, 2012). This design is intended to capture the essence of change in American education. According to this design, "A goal of education in the 21st century is not simply the mastery of content knowledge or use of new technologies. It is the mastery of the learning process. Education should help turn novice learners into expert learners—individuals who want to learn, who know how to learn strategically, and who, in their own highly individual and flexible ways, are well prepared for a lifetime of learning." (NCUDL., 2012) I want to teach students how to learn, not just isolated concepts taught with no exposure to how they would use them in the real world. The three guiding principles for Universal Design for Learning are evident in my project. Principle 1: Provide multiple means of representation (the "what" of learning). Principle 2: Provide multiple means of action and expression (the "how" of learning). Principle 3: Provide multiple means of engagement (the "why" of learning). (NCUDL.2012) I wanted each student to examine the methods of solving quadratic equations and use their own words to lead them to better understanding. They also had the choice in which type of project to complete: poster, booklet or video. Students were allowed to use each other and the internet/technology to better enhance their understanding of underlying concepts prior to completing their project.

As we implement the Common Core State Standards into our classrooms dramatic changes are needed. The project as assigned to students and an exemplary student paper follows:

## Multiple Methods of Solving Quadratic Equations Project

The purpose of this project is to show understanding of how to solve quadratic equations using multiple methods. You should include a step-by-step explanation of how to solve a quadratic equation using multiple methods which include: factoring, graphing, taking square root, quadratic formula, completing the square and table method.
I. You should first choose a presentation method: create a booklet, create a poster, or create a video
II. You should complete the three following sections:
a. Part 1: Solve the equation: $(x-3)^{2}-4=0$

By Taking the Square Root, by Graphing and by the Quadratic Formula
b. Part 2: Solve the Equation $x^{2}-8 x+12=0$ by Factoring, by completing the square and by Table method.
c. Part 3: Create your own equation and solve it using 3 different methods. Be sure to label each method you choose.

## Student Solution of Part One:

Part 1: Solving by square root method

$$
\begin{array}{ll}
(x-3)^{2}-4=0 & \begin{array}{l}
\text { (1) add } 4 \text { to both sides } \\
+4
\end{array} \\
=\sqrt{(x-3)^{2}}=\sqrt{4} & \text { (2) take the square root of both } \\
\text { sides to eliminate the square } \\
x-3= \pm 2 & \text { (3) add } 3 \text { to each side of both } \\
\text { equation possibilities } \\
x-3=2 \\
x=5 & \text { (4) awive at answer } \\
x=5=-2 &
\end{array}
$$

Part 1: solve by graphing


$$
\begin{aligned}
& \text { quadratic }: \frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& \text { formula }: \frac{-1}{}
\end{aligned}
$$

Part 1: solve by using the quadratic formula

## Student Solution of Part Two:

Part 2: solve by factoting


Part 22 Solve by completing the squore

Part 2: compucte by the towle methed


## Student Solution of Part Three:

$$
\begin{aligned}
& \text { quadratic } \\
& \text { formula }: \frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
\end{aligned}
$$

Part 3: solve by using the quadratic formula

$$
\begin{aligned}
& \frac{x^{2}+7 x}{8} x+\frac{10}{2}=0 \quad \text { (1) Plug into quadratic formula } \\
& \frac{-7 \pm \sqrt{(7)^{2}-4(1)(10)}}{2(1)} \\
& \text { (2) simplify } \\
& \text { (3) solve both possible equations } \\
& \text { (4) arrive at answer } \\
& \frac{-7 \pm \sqrt{49-46}}{2} \\
& \begin{array}{l}
\frac{-7 \pm \sqrt{9}}{2} \\
-7 \pm 3
\end{array} x=-5,-2 \\
& \frac{-7 \pm 3}{2}=x \\
& \frac{-7+3}{2}=x \quad \frac{-7-3}{2}=x \\
& x=-2 \quad x=-5 \\
& \text { Part 3: solve by factoring } \\
& \begin{array}{ll}
\frac{x^{2}}{1}+7 x+10=0 & \text { (1) Find what two numbers } \\
(x+5)(x+2)=0 & \text { add to ' } B \text { ' and also multiply } \\
& \text { to ' } C \text { ' }
\end{array} \\
& \begin{array}{lll}
x+5=0 & x+2=0 & \text { (2) put those two numbers with } \\
x=-5 & \text { their } x \text { in factored form }
\end{array} \\
& x=-5 \quad x=-2 \\
& \text { (3) set each quantity equal to } \\
& \text { zero } \\
& x=-5,-2 \\
& \text { (4) solve for } x \\
& \text { (5) alive at answer }
\end{aligned}
$$

Part 3: Solve by graphing


The implementation of the Common Core State Standards also necessitates professionalism. The National Council of Teachers of Mathematics states, "In an excellent mathematics program, educators hold themselves and their colleagues accountable for the mathematical success of every student and for personal and collective professional growth toward effective teaching and learning of mathematics" (NCTM, 2014, p.99). The dramatic change to our classrooms will be better attained when professionals take this responsibility personally and begin to share what they are doing with many others. Good teaching should be shared for the benefit of all teachers and students.

I have gained a tremendous insight into the Common Core State Standards and Standards of Mathematical Practice through my study for my capstone project at Hood College. I believe that taking the time and effort to pay attention to the goals of the implementation of the curriculum gives me a special insight into the process that will serve me well as each course that I teach soon will also be updated to match these standards.

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## Swirling a Vortex of Numbers to See Animal Populations' Futures <br> By Diana Cheng and David Thompson <br> Towson University

How do we prevent species from becoming endangered? What factors cause a species’ population to decline, remain the same, or increase? The purpose of this article is to suggest ways in which middle and high school teachers can incorporate the context of animal populations within science, technology, engineering and mathematics instructional activities. According to the Maryland State STEM Standards of Practice (2012), STEM educational activities should offer students a chance to integrate content from STEM disciplines to investigate and develop solutions for complex global issues. Organizations such as the International Union for the Conservation of Nature (IUCN) (2014), World Wildlife Fund for Nature (WWF) (2014), and the Convention on International Trade in Endangered Species (2002) have convened in the past century to discuss how to preserve animal populations internationally. Identifying and mitigating factors impacting animal populations has indeed been an important real-world, interdisciplinary problem.

## Animal population data in existing curricula

Fortunately, aspects of scientists' and mathematicians’ approaches to analyzing animal populations are accessible at the middle and high school levels, and some classroom activities regarding conservation efforts already have been developed. The Common Core State Standards for Mathematical Practice (CCSSI, 2010) emphasize students’ abilities to use mathematics to describe real-world situations such as those involving animal populations and to use appropriate tools to analyze these situations strategically. The estimation of populations of a given species
via various tagging and statistical sampling methods involves using proportional reasoning (Rubenstein, Craine \& Butts, 2002; Fuentes, 2006a; Fuentes, 2006b). Data regarding endangered species' population size and cost of preservation efforts can be researched and presented using graphs, charts and tables (Muschla \& Muschla, 2006). Determining estimates of geographic ranges in which species live and migration rates involves using trigonometry, statistics, and calculus (Fuentes, 2009a; Fuentes, 2009b; Lacy \& Pollack, 2014).

At the 2014 MCTM conference, we presented a hands-on model of changes to an animal's population. Colored beads represented animals and within each 10 year time period, pairs of animals produced babies and some animals died. We graphed our data, determined the threatened levels based on IUCN classification categories, and used linear, quadratic, and exponential regression to determine which function was the best fit for our collected data. In the spring and fall of 2014, we conducted these activities with in-service teachers taking a graduate level problem solving course as well as pre-service teachers attending a Mathematics Education Club workshop (Thompson \& Cheng, 2014).

## STEM educational activity using Vortex simulation software

This article presents possible extensions to these activities using open-source computer software developed by the Chicago Zoological Society, Vortex 10 (Lacy \& Pollack, 2014) and is available at http://www.vortex10.org/Vortex10.aspx. This software models animal populations and is intended to help biologists understand the effects of natural, demographic, environmental, and genetic forces or random events on these populations (Miller \& Lacy, 2005). For instance, zoologists in Sierra Leone used Vortex software to model the population patterns of the western chimpanzee in an effort to determine which conservation efforts would be more effective at halting its eventual extinction (Carlsen, Leus, Traylor-Hozer \& McKenna, 2012). They
hypothesized that four factors could improve the chimpanzee's population viability: halting of hunting, habitat preservation, prevention of inter-country migration, and expansion of protected geographic areas. Since it was unlikely that all four of these factors could be implemented due to political, physical, and other constraints, it was crucial to determine which of these factors could improve the chimpanzee's population viability the most. After running Vortex simulations with each of these factors being taken into account independently, they found that the best way of protecting the chimpanzee's population was to reduce hunting.


Figure 1. Vortex screen shot highlighting Simulation Input > Scenario Settings tabs.
While using Vortex software, students have the opportunity to mimic the work of biologists to investigate open-ended questions about the parameters that are correlated with or that cause species population trends to be stable, increase, or decrease. Under the "Simulation Input" menu of the Vortex software, the user can input parameters including initial population size, carrying capacity, reproduction rate, percentage of males and females breeding, mortality rate, and number of offspring per female. Disturbances in such parameters affecting populations
can perpetuate, and through positive feedback, can cause further decreases in population size; the types of these disturbances were identified by Gilpin \& Soulé (1986) who called them "extinction vortices," thereby inspiring the name of the Vortex software.

In the first tab of the Simulation Input, in the "Scenario Settings" menu, the user can change the number of iterations and number of populations. For the use of the activities in this paper, we will set the number of iterations to 10 and the number of populations to 2. Based on the data students collect from running simulations and additional outside research, such as researching a species on the IUCN Redlist website, they will also be able to describe reasons why certain parameters impact the simulated population changes.
"Noah's Ark" exploration using Vortex software
An example of an exploration that students can pursue is to explain whether it is truly possible to begin with one male and one female of a species and have them reproduce to repopulate the earth, as is described in the Biblical story of Noah's ark. Some suggested settings in the Scenario Settings menu to help students begin pursuing this question are the following:

- Number of iterations - 10, to have the simulation run ten times
- Number of populations - 1, to have the simulation analyze only one population of the species at a time
- Initial population size -2 , to indicate the one male and one female
- Carrying capacity -1000 , to indicate that the earth can sustain up to 1000 animals

An iteration of this population represents the possible outcome for one species over time. Teachers can discuss with their students what factors (e.g., high reproduction rates and low mortality rates) help a population grow, and what factors are unrealistic to use in this scenario
(e.g., supplementation where additional animals from an external source are introduced into the population).

In Figure 2, we show a sample set of 10 iterations run using a reproduction rate of $100 \%$ and the default mortality rate of $50 \%$. The reproduction rate implies that both animals in the initial population always reproduce in the first year, while the mortality rate implies that there is a $50 \%$ chance that offspring will die. The mortality rate also implies that over a large number of iterations over time, approximately half of the populations will survive and the other half will become extinct. In our simulation of 10 iterations, 6 iterations have populations that survive: as time increases, 5 of these 6 population sizes become close to 1000 , the carrying capacity. Four iterations of the population become extinct: iteration \#3 in year 10, iteration \# 6 in year 14, iteration \#9 in year 1, and iteration \#10 in year 12.


|  |  | Population1 |  |  |  |
| ---: | :---: | :---: | :---: | :---: | :---: |
| Iteration | YrExt | N | GeneDiv | Inbreed | Alleles |
| 1 |  | 909 | 0.6110 | 0.4125 | 3.0 |
| 2 |  | 993 | 0.5827 | 0.4129 | 4.0 |
| 3 | 10 | 0 |  |  |  |
| 4 |  | 1003 | 0.5085 | 0.5045 | 4.0 |
| 5 |  | 948 | 0.1778 | 0.8217 | 2.0 |
| 6 | 14 | 0 |  |  |  |
| 7 |  | 1008 | 0.5175 | 0.4911 | 3.0 |
| 8 |  | 482 | 0.4305 | 0.5851 | 3.0 |
| 9 | 1 | 0 |  |  |  |
| 10 | 12 | 0 |  |  |  |

Figure 2. Graph (left) of population size over 100 years, corresponding table (right): Mortality rate 50\%.

Once students have determined sets of parameters and inputs for these parameters which cause their populations to increase, teachers can then ask their students to justify why these parameters are likely to work. In Vortex, the "Tables and Graphs" tab can be used to create
multiple two-dimensional graphs to represent relationships between combinations of parameters. For example, for each set of parameters chosen, graphs can be produced depicting the probability of survival per year or the probability of extinction per year. Sets of parameters with high potentials for producing surviving populations will yield high probabilities of survival and low probabilities of extinction each year. In Figure 3, we show the graphs of the probabilities of survival and extinction for the species whose parameters correspond to those described in Figure 2.



Figure 3. Graph of species’ probabilities of survival (left) and extinction (right) over 100 years: Mortality rate 50\%.

From Figure 3, we observe that during years 1 through 14, the probability of the species' survival decreases. Each population has a 60\% chance of survival during years 14 through 100; this is consistent with our observation that there were 6 surviving populations. The peak years for probability of extinction are years $1,10,12$, and 14 . These years correspond to years of extinction reported in the table in Figure 2.

We can interpret the data from the graphs in Figures 2 and 3 to conclude that setting the mortality rate at $50 \%$ is too high to keep the population size increasing reliably. We simulated
the population for another 10 iterations using a lower mortality rate of $30 \%$ instead of $50 \%$. We would expect that using an infinitely large number of iterations, that $70 \%$ of them will survive over 100 years and the remaining $30 \%$ would become extinct. When we ran the simulation for 10 iterations, we found that $80 \%$ of the populations survive. The resulting graphs of population size over 100 years, probability of survival over 100 years, and probability of extinction over 100 years are shown in Figures 4 and 5. The graph in Figure 4 shows that 8 of the 10 iterations survive over 100 years. The table in Figure 4 shows 2 of these 10 iterations become extinct (in years 3 and 4). In Figure 5, beginning in year 4, there is an $80 \%$ chance of a population's survival. We can thus conclude through our experiment that using the lower mortality rate corresponds with an increase in the chances of the population's survival.


|  | Population1 |  |  |  |  |
| ---: | :---: | :---: | :---: | :---: | :---: |
| Iteration | YrExt | N | GeneDiv | Inbreed | Alleles |
| 1 |  | 1029 | 0.5589 | 0.4276 | 3.0 |
| 2 |  | 1001 | 0.6453 | 0.3576 | 4.0 |
| 3 | 4 | 0 |  |  |  |
| 4 |  | 1020 | 0.6325 | 0.3412 | 4.0 |
| 5 |  | 994 | 0.5331 | 0.4628 | 3.0 |
| 6 |  | 994 | 0.6555 | 0.3340 | 3.0 |
| 7 |  | 998 | 0.4579 | 0.5501 | 4.0 |
| 8 |  | 1000 | 0.4880 | 0.5150 | 4.0 |
| 9 |  | 1009 | 0.7244 | 0.2775 | 4.0 |
| 10 | 3 | 0 |  |  |  |

Figure 4. Graph (left) and table (right) of species’ population size over 100 years: Mortality rate 30\%.



Figure 5. Graph of species’ probabilities of survival (left) and extinction (right) each year for 100 years: Mortality rate $30 \%$.

## Additional explorations

Additional mathematical activities which we created are posted online at http://tinyurl.com/p673fr7. These activities involve students' investigating parameters or combinations thereof which are associated with a population's change and the rate of a population's change. Common Core State Standards (CCSSI, 2010) which these activities address are listed on the left hand side of Table 2, and our description of how these activities address these standards are listed on the right hand side of Table 1.

| Common Core State Standards: Standards for Mathematical Practice |  |  |
| :---: | :---: | :---: |
| MP2: Reason abstractly and quantitatively. | Students must use their observations to determine which parameters they want to change to create a population that will strive and have a good chance of survival. Students can analyze the Probability of Survival and Extinction graphs developed in Vortex to help with their reasoning. |  |
| arguments \& critique the reasoning of others. | Students are asked to make conjectures about parameters impacting population size and test them using Vortex software. After interpreting data from tables and graphs, they can construct arguments about whether or not the parameters which they chose do in fact cause increasing, decreasing, or stable population trends. |  |
| MP4: Model wit mathematics. | Students are given the opportunity to solve the real-world problem regarding animals' population sizes using the Vortex software. Students may have to experiment with different parameters to adjust their model. |  |
| MP5: Use approp strategically. | Students must strategically determine which quantities to change in the Scenario Settings menu when using Vortex software. To determine a curve of best fit, students must use spreadsheet software. |  |
| Common Core State Standards: Content Standards |  |  |
| HSA.CED.A.2: Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. |  | Students can use data collected to create equations in two variables (population size and time, probability of survival and time, probability of extinction and time) |
| HSF.IF.B.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. |  | Students can identify whether a population size is stable, increasing or decreasing by examining graph which they generate. Students can use the equations they find to determine the year in which a species will become extinct, the year in which a species might reach its carrying capacity, and interpret the asymptotes of the graphs for probability of survival and extinction. |
| HSF.LE.A.1: Distinguish between situations that can be modeled with linear functions and with exponential functions. |  | Students should be able to determine that population models usually can be best modeled using exponential functions. |
| HSS.ID.B.6.A: Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models. |  | Students must determine which type of function best fits their model after collecting data and analyzing $R^{2}$ values corresponding to linear, quadratic and exponential regression. |

Table 1. Common Core State Standards: Standards for Mathematical Practice and Content
Standards which activities described in this article address.

In order to explore the question of "why," additional research is needed beyond the use of
Vortex. To help students better make connections between real world and simulated population
data, we suggest an accompanying research project. Real world data on actual species can be gathered from the IUCN Red List website, http://www.iucnredlist.org/. The IUCN Red List
identifies species that are classified as threatened based on either population reduction, small
overall global population, or both. Students can identify species with steady, increasing, or decreasing population trends; create graphs based on population data presented on the website, and research reasons for the given population trend.

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## Are Math Teachers Biased? Gender Gap Survey

## By Stew Saphier

An article entitled Mind the (Gender) Gap was published in the Fall 2014 edition of the Banneker Banner. This article discussed the gender gap in mathematics between males and females. The thought occurred to us that there may be a bias among people when they learn about such an article. So we decided to perform a survey and try to publish the results if the data collected and the conclusions led to anything of value. Of course, we also felt that almost any result would be of value. And thus you have this article on the results of this survey.

In order to perform this survey in a scientific manner we felt that it had to be performed as a scientific experiment adhering to the five steps of the scientific method. (It was not a requirement that the word 'scientific' be used three times in a single sentence.) The scientific method is sometimes said to have more than five steps, but the five steps we performed are the basis of any scientific method. The five steps of the scientific method are: 1. Create or determine an answer to a question which is your hypothesis; 2. Design an experiment to collect data to answer this question; 3 . Collect the data; 4. Analyze the data collected and determine the results of the data collected; and 5 . Draw a conclusion and determine if your hypothesis was correct.

We will discuss these five steps in order.

## 1. Hypothesis

Our hypothesis is that given an article that discusses the gender gap in mathematics between males and females the majority of people learning about this article will think or guess
that the author of this article is a female. Further, of those that think the author is a male, the majority of them will themselves be a male.

Notice that except for the second part of the hypothesis we believed that the majority would think that the author of a gender gap article would be female regardless of their gender or their profession. The second part of the hypothesis also ignores profession but clearly pays close attention to the gender of the person being surveyed.

## 2. Design an experiment to collect data to answer this question

We decided at first to publish this article in the Banneker Banner without letting the readers know who the author was. We would then ask the readers to send us their thoughts on what the gender of the author was. We also thought that we would survey the participants at the Maryland Council of Teacher of Mathematics (MCTM) conference scheduled after the original publication date of the Banneker Banner. However, this approach ran into a snag. Murphy's Law, or simply the fact that nothing ever goes as planned, did bite us, so to speak. The Banneker Banner was not ready for publication before the conference. So we implemented the survey at the conference and decided not to survey the readers of the article as we wanted to avoid having people 'vote' twice. The survey would catch those reading the article if it was published before the conference as we simply asked them if they knew of such an article. We did not want to ask the readers if they had voted at the conference for multiple reasons.

We now had to create the 'ballot' to hand out at the conference. We clearly needed to know the gender of the person being surveyed - for part two of the hypothesis - and to determine if gender actually made a difference in any event. Of course we had to ask what their guess or thought was as to the gender of the author. We also asked what their profession was to see if that would yield any useful information. And lastly we asked if they knew of any such article as that
could easily bias their answer as to the gender of the author. We decided at this time to discard any ballot where the participant had known of, or read, such an article because of this definitely possible bias.

Here is the ballot that we used:

## Gender Gap Survey

Given that an article is published in a journal discussing the gender gap between males and females in mathematics, we would like your opinion as to whether the author of that article is a male or a female. Please answer anonymously the following questions and drop this 'ballot' in the ballot box.

What is your gender? $\qquad$ Male $\qquad$ Female

What gender do you think/guess the author of this article is? $\qquad$ Male $\qquad$ Female

What is your profession? ___Student ___ Math Teacher ___Other
Have you recently read, or know the author of, such an article? $\qquad$ Yes $\qquad$ No

Thank you for your time in helping us perform this survey.

We then asked for some table space from the MCTM and received space on the MCTM table at the conference. We would put up signs to attract people to come and take our survey including letting them know that this survey would take only about one minute.

We now needed to determine the number of ballots to create so as to get as many responses as possible. Attendance at the MCTM conference has historically been around 800 participants the last several years. We created over 500 ballots and were actually concerned that we wouldn't have enough.

## 3. Collect the data

We now had to collect the data at the conference. This was immediately problematic as we were giving two presentations at the conference and couldn't 'man' the survey table while also speaking. No worries as we expected most everyone would stop and take a one minute survey. We also figured that asking them to vote might peak their curiosity somewhat.

First result: Wow, were we wrong about people stopping. Very few people stopped and took the survey without being asked to do so. It didn't take long to figure this out. When we asked them to take the survey we told them it would take less than a minute and it did, even for those who asked questions about how to fill out the survey. Pleasantly, the majority of people we asked to take the survey did take the survey.

Being optimistic and having over 500 ballots does not make people take the survey. Good thing we are not afraid to approach people to take this survey. Perhaps some might say we are a natural pain in the ass. We had 238 people take the survey, including a few participants before the conference.

## 4. Analyze the data collected and determine the results

First big result from the survey is that it is flawed. We should have had an option that people don't care what the gender of the author is. About four people mentioned that and they are correct, especially since the reasoning of one of them is that they don't think that way as people are people - period.

However, upon further reflection we're glad this option was left out as many people who stated a preference, or guess, probably would not have done so since none of the people taking the survey knew the gender of the person writing the article and simply saying "I don't know" is an easy out. This way they had to make a choice, or create the third option - which only six people did by not answering the question or checking an "I don't know" box. So this is one case where a flawed survey is actually better.

One thing to notice from the data collected is that the total population of respondents is only 238 people. A number that is not quite sufficient to have very accurate results. A population of 1,000 would be much preferred. But this is the population we have and must therefore deal with. We must also therefore acknowledge that any results given may not be completely accurate.

Herewith the data that was collected:


## Table 1: Gender Gap Survey Data

From the data collected it is clear that by more than a three to two margin (110 or $62 \%$ to 68 or 38\%) of those responding AND giving us their guess as to the author's gender, they thought the author would be a female. Of those guessing female, 85 or $77 \%$ are themselves female and 25 or $23 \%$ are male. But since $80 \%$ of the respondents are female, this is within a margin of error of variance and thus no conclusion about gender of the respondents can be drawn from this result. However, of those guessing that the author would be a male, 59 or $87 \%$ are female and 9 or $13 \%$ are male. These are different percentages from the guesses of female and appear to be outside a
margin of error for a population of $80 \%$ females. Thus it would appear that females may be more apt to think the author is a male, but not by much.

How much does the profession of the respondent play a role in these results? First, we must notice that the students in this survey, since it was taken at the MCTM conference, are all student teachers. Indeed, we were asked a few times as to how a person who is a student teacher should respond to this category. These results vary wildly. Let's first look at the males. Of the five male students responding three, or $60 \%$, thought the author would be a male. Two, or $40 \%$, thought the author would be a female. While the percentages may at first glance appear outside a margin of error, they are not as there are only five respondents. Thus, nothing learned here. Of the 20 male math teachers giving a guess as to the gender of the author, 6 , or $30 \%$, thought the author would be a male. The other 14 , or $70 \%$, thought the author would be a female. While only 20 people here, this appears to say that male math teachers lean toward an author of female.

Perhaps one of the most interesting results, and clearly outside a margin of error (if not for the low number of respondents - nine) is that all 9 , or $100 \%$, of the male respondents in other professions thought the author would be a female. This is even more interesting when looking at the females in the other professions as that result is not nearly the same. In that case 9 out of 16, or $56 \%$ thought the author was a female. The other 7 , or $44 \%$, thought the author was a male. With only 16 respondents this is not a result worthy of note.

Let's take a frivolous look at this $100 \%$ of 9 statistic. The major sport that undoubtedly keeps the most statistics is Major League Baseball. As we're writing this the World Series this year (2014) is going into a game seven. Of the previous nine World Series that went to a game seven all nine were won by the home team thus giving the Kansas City Royals (the home team) the clear, and maximum, statistical advantage - $100 \%$ to $0 \%$. [SB Nation on the Web:
www.sbnation.com; October 29, 2014; Busbee, Grant] And, of the previous 60 teams to trail 3 to 2 in a World Series, 36, or $60 \%$ of them won game 6, as the Kansas City Royals did. Of those 36, 19, or $53 \%$, also won game seven and thereby the World Series. [ESPN.go.com; Stats \& Info; October 29, 2014.] Thus, the advantage still goes to the Kansas City Royals, but by a much smaller amount, $53 \%$. And since this is only one more than the 18 of 36 that would have made it a $50-50$ proposition, this is the minimal statistical advantage. I think this demonstrates the futility of making any strong conclusion based on only nine data points, even when all nine data points have the same value. And just to be clear, the above was written well before the game. Then just before the game the three TV commentators asked to predict the outcome of the game all thought that Kansas City would win. [Televised broadcast of the game on Fox] Since we have seen the potential meaningless of nine data points agreeing, obviously three data points agreeing has even less meaning. And for those who don’t know, the San Francisco Giants won the game.

Now we could keep going, so to speak, and come up with more analysis and results, but the numbers tend to be too small for any meaningful conclusion. So we move on to step five.

## 5. Draw a conclusion and determine if your hypothesis was correct

The first conclusion we can draw pertaining to our hypothesis is that from this data the first part of our hypothesis (that most people will think the author is female) is correct. The second part of our hypothesis (that of those thinking the author is a male will themselves be male) is incorrect. Oh well.

## References

Busbee, Grant; SB Nation on the Web: www.sbnation.com; October 29, 2014
ESPN.go.com; Stats \& Info; October 29, 2014
Televised broadcast of game seven of the 2014 World Series on Fox

The Maryland Council of Teachers of Mathematics (MCTM) is the professional organization for Maryland's teachers of mathematics. Our members represent all levels of mathematics educators, from preschool through college. We are an affiliate of the National Council of Teachers of Mathematics (NCTM). Our goal is to support teachers in their professional endeavors and help them to become agents of change in mathematics education.

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> MCTM Mission Statement: The MCTM is a public voice of mathematics education, inspiring vision, providing leadership, offering professional development, and supporting equitable mathematics learning of the highest quality for all students.


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