# The Banneker Banner

Maryland Council of Teachers of Mathematics

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The individuals below have given their time and expertise to read and review manuscripts submitted for this edition of the *Banneker Banner*. We are very grateful for their help.

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## *Banneker Banner* Submission Guidelines

The Banner welcomes submissions from all members of the mathematics education community, not just MCTM members. To submit an article, please attach a Microsoft Word document to an email addressed to strickland@hood.edu with "Banneker Banner Article Submission" in the subject line. Manuscripts should be original and may not be previously published or under review with other publications. However, published manuscripts may be submitted with written permission from the previous publisher. Manuscripts should be double-spaced, 12 point Times New Roman font, and a maximum of 8 pages. APA format should be used throughout the manuscript with references listed at the end. Figures, tables, and graphs should be embedded in the manuscript. As the Banner uses a blind review process, no author identification should appear on manuscripts. Please include a cover letter containing author(s) name(s) and contact information as well as a statement regarding the originality of the work and that the manuscript is not currently under review elsewhere (unless accompanied by permission from previous publisher). If electronic submission is not possible, please contact the editor to make other arrangements. You will receive confirmation of receipt of your article within a few days, and will hear about the status of your article as soon as possible. Articles are sent out to other mathematics educators for anonymous review, and this process often takes several months. If you have questions about the status of your article during this time, please feel free to contact the editor. Please note that photographs of students require signed releases to be published; if your article is accepted, a copy of the release will be sent to you and it will be your responsibility to get the appropriate signatures. If you would like a copy of this form at an earlier time, please contact the editor.

Readers who have developed successful classroom activities are encouraged to submit manuscripts in a format suitable for immediate use in the classroom. Submissions should help students understand mathematics or help teachers teach mathematics from either a conceptual or procedural instance while modeling effective pedagogy and addressing at least one Maryland College and Career Readiness Standards/Common Core Content and/or Practice Standards. A successful lesson or activity is one that is enjoyable to teach, that works well with students, that other teachers might adapt for use in their own classroom; and that is centered in developing students problem-solving and discourse

Student Activities can be for any grade level and should be in final format. Use one-inch margins, Time News Roman font, and a maximum of 4 pages. Authors should include a brief description of theirs and their students' experiences during the implementation as well as of insights gained from it. Any citation should follow APA Style (6th edition). The required list of references may not count toward the four-page limit of this section. Submissions will be sent out for peer review. Prospective authors should submit manuscripts of "Student Activities" to prof.lima@gmail.com .

# Message from the President

Jennifer Novak

*Learning never exhausts the mind."*

-Leonardo da Vinci

As we enter a new school year, we make several plans for teaching and extra-curricular activities. It's also critically important that we make time to focus on our own continued learning as educators.

This Fall 2016 edition of *The Banneker Banner*, the official journal of the Maryland Council of Teachers of Mathematics, provides the latest research for the members of MCTM to support our professional learning. A great thanks to the editor of this edition's *Banner*, Tricia Strickland, and the Publications committee, who have volunteered countless hours to review submissions and assemble this publication.

As I begin my seventh year on the MCTM Board and second year as President, I am excited for the strides we have made to support teachers across the state. We are continuing to look for creative, low-cost ways to provide high-quality professional learning to our dedicated members.

This past year has been no exception. In addition to our annual conference, MCTM sponsored two regional conferences this spring: a Central Regional conference in Columbia, MD and an Eastern Shore Regional conference in Wye Mills, MD. We continued to host an Elementary Summer Academy and, for the first time, offered a Middle School Summer Academy. Last winter, we also sponsored our first Periscope panel discussion focused on math coaching. These events would not have been possible without the commitment and creativity of our amazing MCTM Board of Directors. We are always looking for volunteers to assist with the planning and on-site support for these events. If you are interested in volunteering, e-mail your MCTM region representative.

I hope you enjoy this latest installment of *The Banneker Banner* and make use of our upcoming professional learning opportunities, including our Annual Conference on Saturday, October 22, 2016. To get the latest MCTM news, visit https://www.marylandmathematics.org/. We have worked to update our website to provide timely information and resources to our members and will be adding some "members only" perks soon.

As always, your MCTM Board and I are here to serve you. Please contact us at any time to help us best meet your needs.

Jennifer Novak

# Using Interactive Student Notebooks (ISNs) To Increase Achievement in At-Risk Algebra Students

*By Nicole Y. Henry, Frederick County Public Schools*

Research reported in this paper was completed in a capstone project for the master's degree in mathematics education in the Graduate School of Hood College.

First-year Algebra has long been thought of as a gateway course to upper level mathematics and science courses. Over the last 50 years, the educational importance of Algebra has continued to grow. Today, Algebra or a course in algebraic concepts is a high school graduation requirement in thirty-five states in the United States and is at the "core" of the Common Core State Standards for Mathematics (CCSSM). As states and school districts are being pressured to graduate students who are college and career ready, the importance of Algebra cannot be denied.

The education reform movement argues that *Algebra for All* is crucial in creating students who are college and career ready (Eddy, 2014). Educators need to ensure that they are teaching Algebra so that every student is able to understand the algebraic concepts and that the abstractness necessary for learning such concepts does not filter out those with disabilities, poor previous math performance or career goals (Allensworth, 2009). If one agrees with Lynn Arthur Steen, a professor of Mathematics at St. Olaf College, that Algebra is "an invaluable engine of equity" and "a passport to advanced mathematics" (1999), then it is imperative that educators make Algebra accessible to all students, and ensure all mathematical experiences, especially high school ones, are equitable.

Students can find success in first-year Algebra courses when teachers remember that students from diverse backgrounds have diverse needs. The actual content of the course being taught may actually be less important than pedagogy and the relationships formed in the classroom (Allensworth, 2009, p. 383). Activities that foster independence seem to help students engage in challenges they may have previously thought out of reach.

A teacher's role in working with at-risk Algebra students, however, does not end with engagement. The *Growth Mindset* movement grew out of the idea that students' attitudes toward their own abilities (their mindset) are connected to both their motivation and achievement. Students who believe that their abilities can be developed do better than those students who believe that their abilities are fixed (Dweck, 2015). Dweck encourages teachers to include the concept of growth mindset in their pedagogy. She advises, "Teachers who understand the growth mindset do everything in their power to unlock that learning" (Dweck, 2015, p.2).

Note-taking systems can be tools to help increase student achievement and engagement in the Algebra classroom, especially for at-risk students. Caroline Wist, of The College of William and Mary says, "The process of taking notes actively engages students in the learning process which increases comprehension" (2006, p.30). A note-taking tool created specifically to increase student growth is the Interactive Student Notebook (ISN). The ISN was initially developed in the 1970s by teachers at Aragon High School in San Mateo, California as a way to engage and motivate Social Studies students. Teachers' Curriculum Institute (n.d.), a publishing company run by teachers, heard about the successes these teachers were having with the notebooks, and worked with them to include the ISN as part of the TCI approach to teaching and learning, which aims to combine "great content, meaningful technology and interactive classroom experiences" (Teachers Curriculum Institute, n.d.) to ensure all students are successful.

When TCI adapted the Aragon High teachers' notebook idea they wanted students to be able to create unique, personal records of their learning (Teachers Curriculum Institute, n.d.). Essential parts of each lesson involving the notebook are:

1. Preview Assignments – connecting the day's lesson to prior knowledge;
2. Graphically Organized Reading/Lecture Notes;
3. Processing Assignments – synthesizing what they have learned.

The ISN allows students to become actively involved in their own learning. From the Preview Assignments which access students' prior knowledge, through active notetaking, to Processing Assignments which often involve graphic organizers, students are encouraged to be autonomous thinkers and access their creativity. Students who work to keep and use their notebook, can see that their efforts help them succeed.

While teachers may disagree about the type of notebook to use, most teachers who use ISNs agree on the basic notebook set up which employs a right-left technique. The right side is the input side, where reading and lecture notes and other teacher defined activities are completed and the left side is the output side, where students independently process the information on the input side (Chespro, 2006). Output pages demonstrate that the students understand the input material.

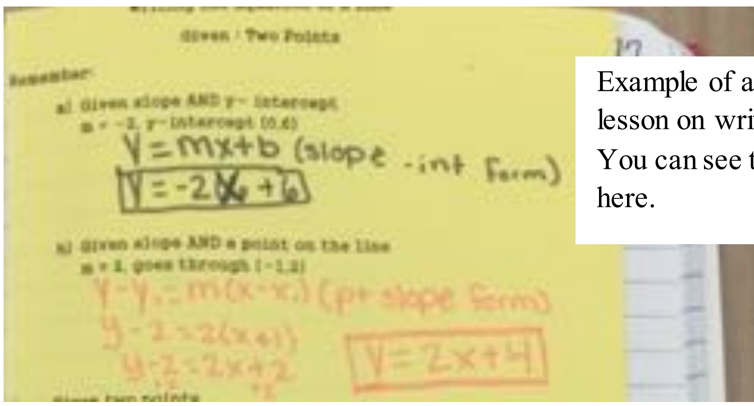
Teachers who have used Interactive Student Notebooks in their classrooms have varied experiences and report different outcomes. Brad Lewis, a journalism teacher who was first introduced to the ISN while he was a long-term substitute teacher, realized that once he adapted the ISN to his own teaching style, It became, "much easier to use and understand" (2013, p. 32).

He says the ISN "enables students to discover the rewards of diligence and of discipline" (2013, p. 32). Robert Chespro, who uses the Interactive Student Notebook in his science classes, has also adapted the notebook to his own style and feels that this adaptation is an essential step for teachers to find success with ISNs as well (2006). Although most of the scholarly work on ISNs has dealt with the secondary science classroom, teachers using the Interactive Student Notebooks in the mathematics classroom have had similar results. Jennifer Smith, who hosts the blog *4mulaFun*, has had so much success using the ISN that she offers workshops across the country on how to best use them. She says that the Interactive Student Notebook is "not only a resource for students, … but a tool to engage students' minds and set the foundation for content" (Smith, 2014).

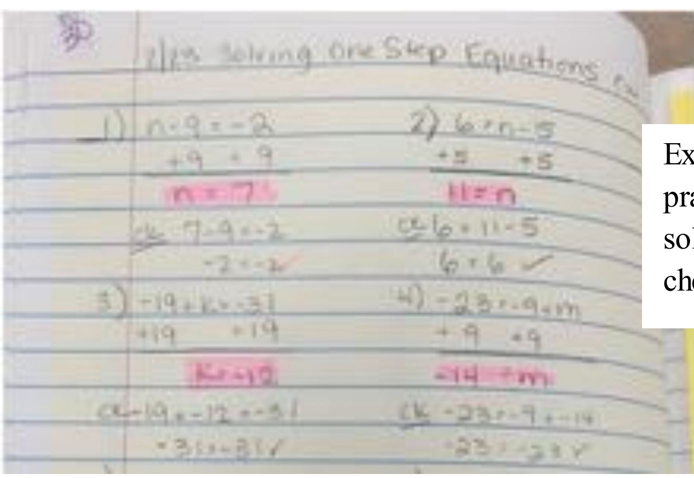
In the summer of 2013, I began reading numerous educator blogs, as I looked ahead to the beginning of a new school year as a high school math teacher. The thing that most piqued my interest was using Interactive Student Notebooks to help struggling students learn and retain math content. I read all that I could about ISNs that summer and decided to use them for my Geometry classes. As the year went on, I was busier, and didn't plan out my pages regularly so students never really took ownership of their notebooks. The notebooks were still used by the students, but when I informally surveyed them, the feeling was that they had not been helpful. I decided not to use the ISNs the next year and took time to reflect on whether their use was worth the time and effort put in by both me and my students.

Based on my research into ISNs, I felt strongly that they could benefit my level 1 Conceptual Algebra students. These students are the lowest level Algebra students at the high school where I teach, so based on my research, I set out to incorporate ISNs into these classes. I planned on using the first semester class as a baseline, reflect on my use of the ISN and modify what needed to change.

At the beginning of the year, I purchased each student a composition notebook. I have found composition books to be superior to spiral notebooks as the pages do not come out as easily. I incorporated the ISNs daily, so that it would become an important part of the regular routine of the class. Students have their ISNs on their desks when they come into class every day and they are used throughout the class period. Here are some examples of pages we have completed.




Example of a right hand notes page - a lesson on writing the equation of a line. You can see the student's use of color here.




Example of a left hand guided practice page. The student is solving one-step equations and checking her solutions

Student notes are in one place, and they can refer to their notebook whenever needed. I keep my own "teacher notebook" so that students always have a place to get information when they are absent from class for any reason.

As anticipated, not all of my students have positive attitudes toward the use of the Interactive Student Notebooks. Most of the students, however, could agree that the notebooks were beneficial to them. Sarah, a student in my first semester class said, “The notebook helped me stay organized,” while Ben said, “It wasn’t a pain to take notes – Mrs. Henry helped us …the organizers helped me study what was important before the tests.”

As I reflect on the use of the Interactive Student Notebooks in my Conceptual Algebra classes this year, I am glad that I began using them, and I believe that their use was a benefit to the students. Students improved in both engagement and achievement. Use of the ISNs helped some students increase their grades, while others were able to become more organized learners.

Comparing first quarter grades of each class – the first semester class I used as a baseline, to the second semester class where I made modifications, will help me make decisions as I go forward. Numerical grades are assumed to be percentages. The side by side boxplot in figure 1 shows the distribution of the data for the semester 1 class and the semester 2 class. The upper box plot shows that the range of the students’ (semester 1) quarter 1 grades is 46 - from a 53 to 99. Although the lowest grade is a 53, almost 75% of the grades are above 70. The median quarter 1 grade was 77, meaning that the upper 50% of scores was between 77 and 99. The lower box plot in figure 1 shows the distribution of the data for the semester 2 class. There is a slightly wider range of scores with this class than with the semester 1 class – from 50 to 100. The semester 2 class has a much higher median score (87) than the semester 1 class. This causes the lower 50% of the scores to have a much wider range. This is encouraging, as the upper 50% of the scores, though narrower, is from 87 to 100.

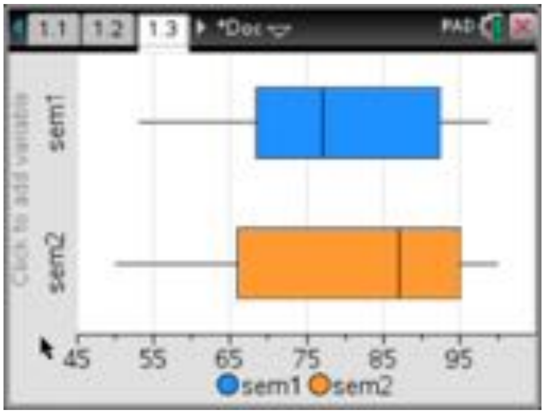



Figure 1

Box Plot comparing Semester 1 to Semester 2

It is important to note that this is not a completely valid statistically study. I am comparing two different sets of students who bring to class completely different sets of abilities and previous knowledge. In addition, there are many other lurking variables including gender, family situation and work ethic that confound the outcomes. However, as I look at the data I can't help but feel that the use of the notebooks worked as a positive force in my classroom. I will definitely use the Interactive Student Notebooks in my Conceptual Algebra classes next year. I found the planning and executing stimulating and was pleased seeing students use them as a resource.

Now that Algebra is a graduation requirement in the majority of the United States, we need to be cognizant of methods we use to prepare students to master algebraic concepts that are important for them to be college and career ready. Educators need to be aware of all students' strengths and weaknesses and engage them in meaningful and equitable coursework so that they have access to opportunities in the future. Ensuring students have good note taking skills and are able to use and reflect on those notes becomes increasingly important as students' progress through high school. Interactive Student Notebooks can help students to actively take and interact with their class notes. Teachers who have used these ISNs have had positive results both with engagement and achievement. The Interactive Student Notebooks help foster student independence and use a variety of strategies to become actively involved in their learning (Teachers Curriculum Institute, n.d.).

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# Building a Bridge… of Questions

*By Michelle Harriger, Montgomery County Public Schools*

Research reported in this paper was completed in a capstone project for the master's degree in mathematics education in the Graduate School of Hood College.

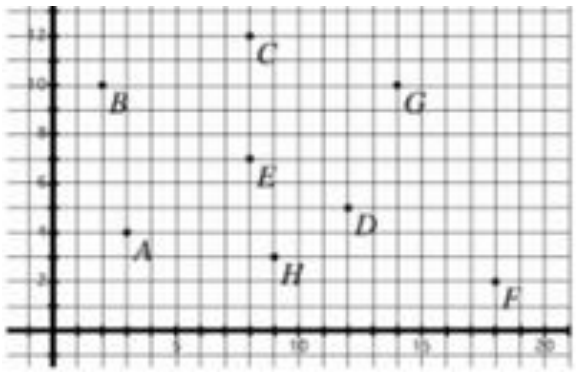
We've all been there. You think a lesson is going well, students are making progress through a topic, discussing with one another, when all of a sudden…Splat. Epic lesson failure. From around the room, you hear a chorus of "Teacher! Teacher! I need help!" There's no way to get to everyone individually in the time allotted. In all likelihood, the thought, "Direct instruction was so much easier… maybe this topic just can't be taught via the Common Core's Standards for Mathematical Practice" crossed your mind. To give in to this notion, however, would be a shame, as it robs our students of the opportunity to make connections between mathematical concepts as well as creating their own path to a solution. In time spent planning and revising lessons with my Professional Learning Community (PLC), I've found that the most likely cause of the Epic Lesson Failure centers on the questions posed. Specifically, if students are asked to make too big a leap on their own, they fall. The solution, therefore, is to create an intermediate question just prior to the point of failure, to provide a series of questions that are challenging enough for students to remain engaged, but simple enough that students can make each leap on their own. Such a series of questions will necessarily be different than the series of questions found in textbooks or lessons completed via direct instruction.

Here, I've provided three examples of a series of questions that failed, miserably, on the first attempt, but were successful the following year once intermediate questions were added.

**Functions that Perform Transformations in the Coordinate Plane:**

The goal of this lesson is to meet CCSSM Geometry Congruence Standard 2, in particular, that students should be able to "describe transformations as functions that take points in the plane as inputs and give other points as outputs." Our original question series was as follows:

Question 1: Perform transformation $T$ on a figure in the coordinate plane, listing the coordinates of the input and output.




Translate point $D$ five spaces to the right. Record the coordinates of the pre-image and image in the table below.

Perform the same translation on point $G$. Record the coordinates of the pre-image and image in the table below.

Now choose your own point, and perform the same translation. Record the coordinates for the pre-image and image in the table.

| Input | Output |
|---|---|
| $D(\_\_\_\_, \_\_\_\_) \to D'(\_\_\_\_, \_\_\_\_)$ | |
| $G(\_\_\_\_, \_\_\_\_) \to G'(\_\_\_\_, \_\_\_\_)$ | |
| | |

Question 2: Look for a pattern between the coordinates of the input and the coordinates of the output.

Question 3: Write a rule that will take any point (x, y) to its new location under the same transformation.

> Generalize your findings into a rule that will take any point $(x, y)$ as the input. If $(x, y)$ undergoes the same translation, what will be the coordinates of the output?
>
> $$(x, y) \rightarrow (\_\_\_\_\_\_\_\_,\ \_\_\_\_\_\_\_\_)$$

I still can't, for the life of me, figure out why students who could beautifully describe the change in x- and y-coordinates under a given transformation were so befuddled when asked to apply this pattern to the point (x, y), but they were. One additional question, however, made a remarkable change. Our updated series of questions is as follows (the intermediate question is italicized):

Question 1: Perform transformation $T$ on a figure in the coordinate plane, listing the coordinates of the input and output.

Question 2: Look for a pattern between the coordinates of the input and the coordinates of the output.

*Question 3: Apply this pattern to the point P(482, 167). What will be the coordinates of P' under the same transformation?*

> Point P (not shown in the figure) has coordinates $(482, 167)$. If point P were to undergo the same translation (translate five spaces to the right), what will be the coordinates of the output P'? Use the data from your table to help you determine your answer.
>
> $$(482, 167) \rightarrow (\_\_\_\_\_\_\_\_,\ \_\_\_\_\_\_\_\_)$$

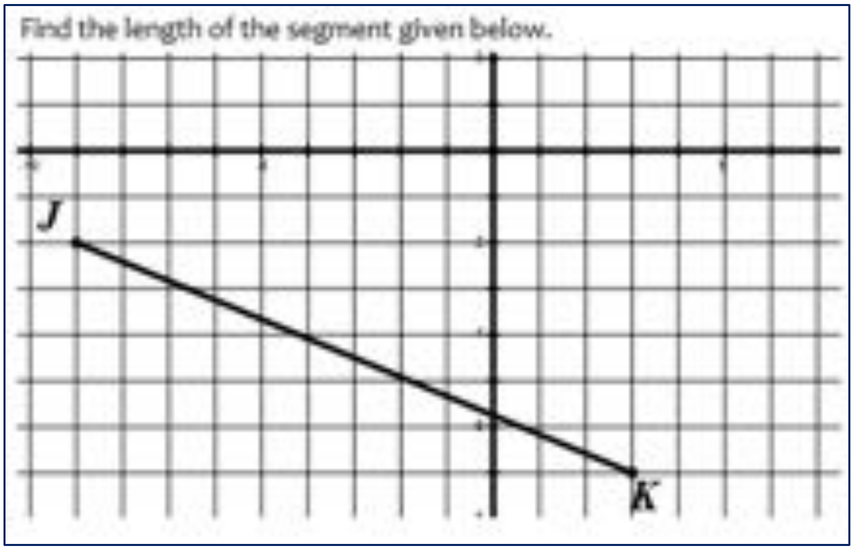
Question 4: Write a rule that will take any point (x, y) to its new location under the same transformation.

The change was remarkable! Even the following year with a completely new group of students, our classes were able to work through all four tasks with little intervention.

**Equations of Circles**

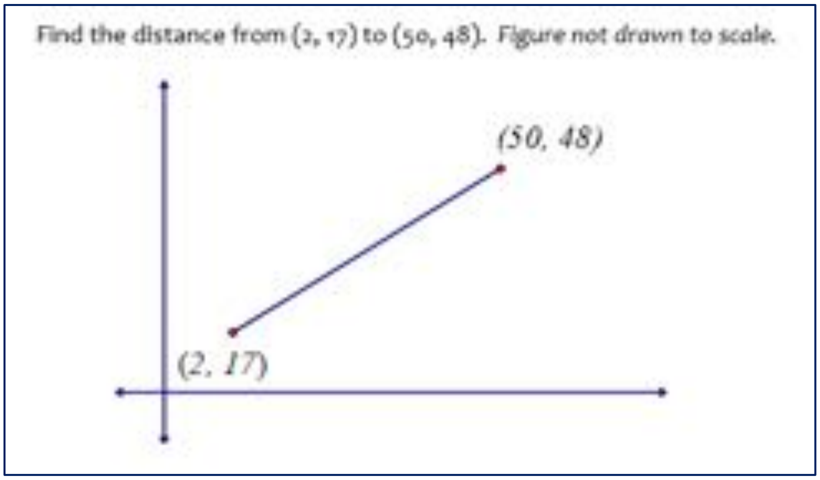
The goal of this lesson is to meet CCSSM Expressing Geometric Properties with Equations Standard 1, specifically, to "derive the equation of a circle given center and radius using Pythagorean Theorem." The lesson originally posed the following series of questions:

Question 1: Find the distance between two points (shown on a coordinate plane).



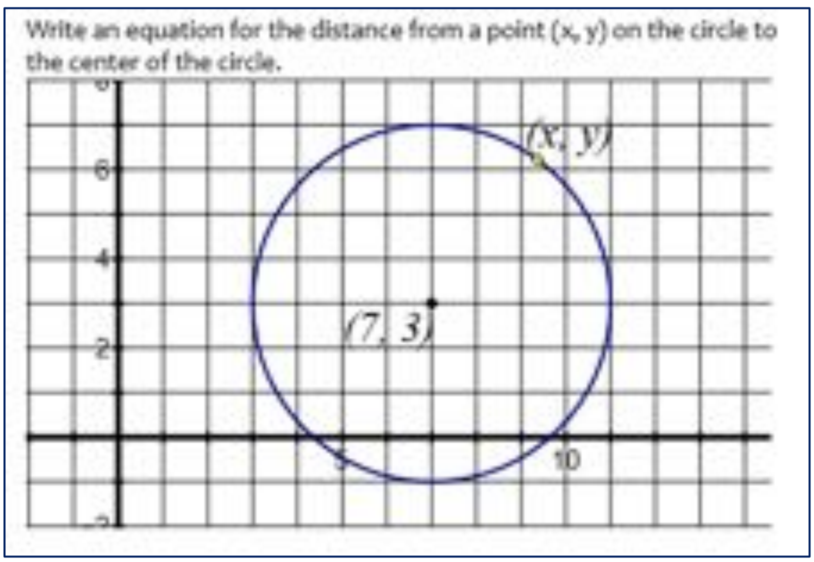

This question established sketching a right triangle and using the Pythagorean Theorem as a method for finding the distance between two points. The legs of the triangle were both easily countable on the grid.

Question 2: Find the distance between two points whose coordinates were disparate enough to make counting the length of the legs inconvenient.




This question established the length of the horizontal and vertical legs as the difference in the x-coordinates and the difference in the y-coordinates, respectively.

Question 3: Write an expression for the distance between two points: the center of the circle and a point (x, y) on the circle.




Question 4: Given the radius and center of a circle, write an equation showing the distance between two points (the center of the circle and a point (x, y) on the circle) is the radius of the circle.

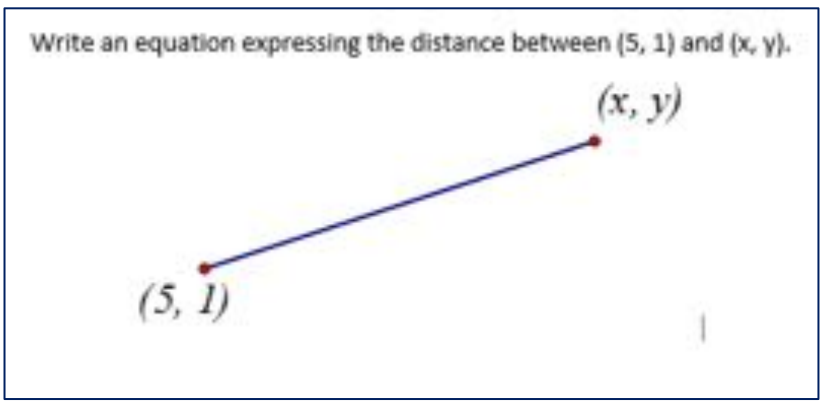
Classes finished questions 1 and 2 in under 15 minutes, and found them easy. We began question 3 with 20 minutes left in the period. We never made it to question 4. A day later, we finally made it through, but we were now a day behind.

The following year, we tried the following adjustment (intermediate question italicized:

Question 1: Find the distance between two points (shown on a coordinate plane).

Question 2: Find the distance between two points whose coordinates were disparate enough to make counting the length of the legs inconvenient.

*Question 3: Write an expression for the distance between a given point and a random point (x, y).*




Question 4: Write an expression for the distance between two points: the center of the circle and a point (x, y) on the circle.

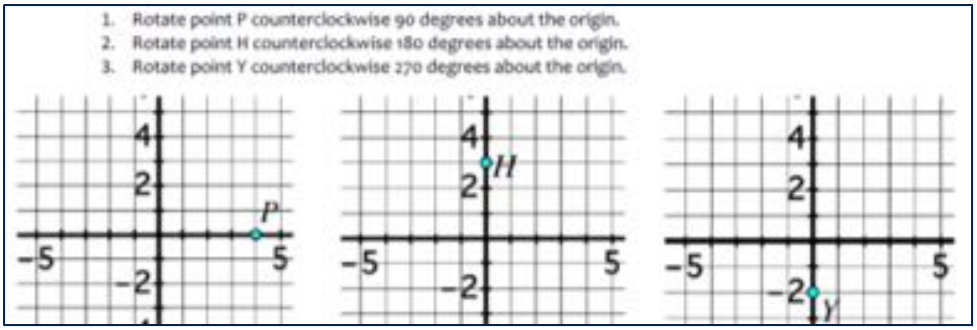
Question 5: Given the radius and center of a circle, write an equation showing the distance between two points (the center of the circle and a point (x, y) on the circle) is the radius of the circle.

Once again, an additional question made all the difference. The entire series of questions was completed in one class period, no hair-pulling necessary.

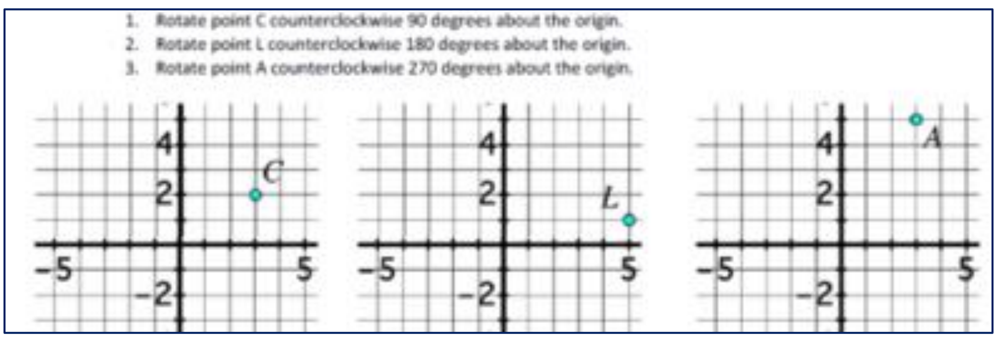
**90-degree Rotations in the Coordinate Plane about the Origin:**

Our lesson on performing rotations (addressing a portion of CCSSM Congruence Standard 2) began by defining rotations carefully. We then assigned the following tasks:

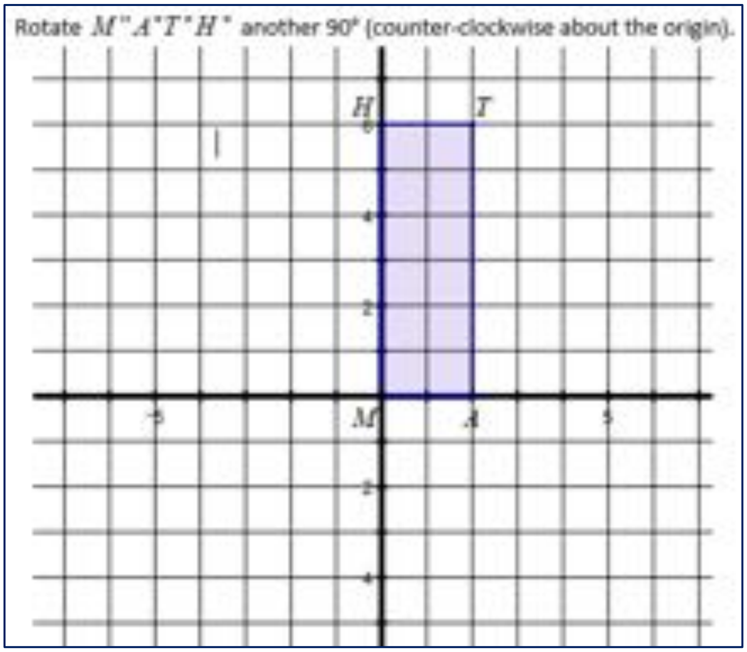
Question 1: Rotate a point given on either the x- or y-axis through some multiple of 90 degrees in a given direction about the origin.




Question 2: Rotate a point in one of the quadrants through some multiple of 90 degrees in a given direction about the origin.




That students got stuck with question 2 was not entirely a surprise, but the degree to which our classes got bogged down was surprising. Despite knowing the definition of rotations, having access to compasses, and being able to correctly state the center and number of degrees of rotation needed to complete the problem, students found it very difficult to visualize when to “stop” rotating. A colleague came up with the questions below to try and help students see where each multiple of 90 degrees brought a point:




This question provided an "a-ha" moment for a number of students, so the following year, we included it between questions 1 and 2:

Question 1: Rotate a point given on either the x- or y-axis through some multiple of 90 degrees in a given direction about the origin.

*Question 2: Rotate rectangle MATH 90° counter-clockwise about the origin*

Question 3: Rotate a point in one of the quadrants through some multiple of 90 degrees in a given direction about the origin.

Once again, the intermediate question saved the day! By the end of this series of questions, virtually all of our students were rotating points correctly.

The next time you're faced with a lesson that comes to a screeching halt, take some time to consider an intermediate question. First, look at the series of questions posed leading up to the point at which the lesson stalled. Second, consider the intellectual leap that must be made to get from the question just preceding the stall to the question that caused it. Third, create a new question to pose between these two, which asks students to make part of the leap. This is a terrific time to consult your course team—they can help you think of questions to ask, help you determine whether the intent of your question is clear, and offer other strategies. Once you've revised your question series, it's time to test it out. Make note of how the revised lesson went. Did all run smoothly? Keep your series of questions for next year. Are there still some kinks to smooth over? Make note of those as well. Creating a successful lesson takes time, care, and in all likelihood a few revisions. With careful notes, your team can create a series of questions that will lead to substantially better lesson plans.

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# Grade 4-6 students' meanings for unknown addends: Implications for algebra

*By J. Matt Switzer, Andrews Institute of Mathematics and Science Education*
*Texas Christian University*

*Sally, a fourth grader, was asked what numbers the shapes in $\square + \Delta = 12$ could be.*
***Interviewer**: So what do the shapes mean for this one?*
***Sally**: Different numbers.*
***Interviewer**: So what numbers can I put in there that would make that work?*
***Sally**: Different numbers, eight and four. I can't do six and six because they represent different numbers. Ten and twelve, eleven and one, five and six, wait, five and seven.*

Students, such as Sally, begin generalizing meanings for formal, informal, and idiosyncratic mathematical symbols as soon as they begin using the symbols. Drawing on their prior knowledge and experiences, students' generalizations may or may not be the same as the teacher's or the generalization we intend for our students to develop. In this article, I describe findings from a research study exploring grade 4 – 6 students' meanings for formal and informal representations of unknowns in double unknown addend tasks, compare these findings to common student misconceptions for letters-as-variables in algebra classes and beyond, and provide recommendations for assisting students in transitioning to the use of letters-as-variables and their conventions of use as they move to mathematics in the middle years.

**Students' Experiences with Informal Representations of Variables and Unknowns**

The National Council of Teachers of Mathematics (NCTM) suggested that Pre-K-2 students "use concrete, pictorial, and verbal representations to develop an understanding of invented and conventional symbolic notations" (National Council of Teachers of Mathematics, 2000, p. 90). Likewise, the Common Core State Standards for Mathematics (CCSSM) expects students in first grade to represent unknowns in all positions of addition and subtraction problems "by using objects, drawings, and equations with a symbol for the number to represent the problem (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 15). In Grades 3-5, NCTM further suggested that students "represent *the idea of a* variable as an unknown quantity using a letter or a symbol" (NCTM, 2000, p. 158, emphasis added). These suggestions reflect the typical progression from informal representations of variables and unknowns to conventional letters found in elementary mathematics curriculum materials. The CCSSM does not include "variable" until grade 6, potentially perpetuating findings by Knuth, et. al. (2005) that "knowledge of the concept of variable may be somewhat fragile, particularly among 6$^{th}$-grade students" p. 75.

Philipp (1992) defined variables as "consisting of a symbol standing as a referent for a set consisting of at least two elements" (p. 557). When applying this definition

> [e]ven the literal symbol $x$ in the statement $x+3=7$ is a variable, because $x$ represents any of the elements of the set in the unstated but implicitly assumed domain, be it the real numbers, the rational numbers, the integers, the natural numbers, as so forth (p. 577).

Usiskin (1999) noted the multiple definitions, referents, and symbols possible for variables and how "conceptions of variable change over time" (p.7). The researcher's purpose is to attend to *student generated* generalization(s) for various as opposed to promoting a particular definition, representation, or conception of variable. Specifically, representations of unknown addends in two unknown addends tasks with a given sum (e.g. $a+b=12$ and $\_+\_=7$).

As noted by Usiskin (1999) and Philipp (1992), a specific definition for and representation of variable is not universally established. Further compounding the issue differentiating between variables, unknowns, and placeholders is students' interpretation of the various representations symbols. In the absence of adopting algebraic conventions for variables, and the concept of variable, the questions of whether a symbol standing for a number(s) is an unknown quantity, variable, or place holder for a single value is relative to the students' various perspectives.

Student generalizations for the variety of representations of variables and unknowns, hereby referred to as unknowns, and whether they contribute to or hinder their understanding of letters-as-variables is less well known. The author reviewed three commonly used elementary mathematics textbook series (*enVisionMATH*, 2009; *Investigations in Number, Data, and Space*, 2008; *Mathematics: The Path to Math Success!*, 1998) and found a variety of representations for variables and unknowns including blanks, letters, shapes, and words. Typically, tasks in which these representations were used contained a single variable, or unknown, where the solution was a single value (Knuth et al., 2005). For such tasks, students are typically expected to determine the value for the variable, or unknown, or to substitute a given value for a variable or unknown into an expression and then evaluate the expression.

Shapes were a common staple throughout much of the elementary textbooks used by the students in this study. For example, a light blue shaded square was commonly used to represent unknowns (*enVisionMATH*, 2009). However, the text also used squares, see figure 1, in combination with letters. The following task was included on a unit assessment for $5^{th}$ grade.

$$y \times 9 = 72$$
$$y \times 9 \div \square = 72 \div \square$$
$$y = \square$$

**Figure 1: 5th Grade Task with Square as Placeholder**

The use of the square (□) in the above problem illustrates how such tasks may contribute to student difficulties working with and understanding variables in algebra and beyond. First, substituting *any* non-zero real number for the two squares in the second equation will result in an equivalent equation. However, the authors' implicit intention is that the squares in the second equation are the single value that will result in the specific equivalent equation $y = \square$. Each □ is intended to be a place holder for 9, resulting in $y \times 9 \div \boxed{9} = 72 \div \boxed{9}$ in order to isolate the *y* on the left side of the equation, which is not explicitly included in the textbook's provided solution. The structure of the problem suggests that substituting a value other than 9 for the squares in the second equation is incorrect.

Second, the square (□) in the third equation, $y = \square$, takes on the value of 8 from $72 \div \boxed{9}$, a *different* value from the squares in the second equation. The squares do not maintain the same value throughout the three equivalent equations in the solution as would be the case if applying algebraic conventions. The □'s are used as placeholders for the student to "fill in" as opposed to a mathematical symbol representing the same number. Whether students would make such a subtle distinction is unclear and likely not assessed.

The textbooks also used blanks to represent unknown quantities, such as 18 – 9 = __ and 3 + __ = 7. As with the square in the previous example, it is unclear if the blank is intended as a place holder to be "filled in" or as a symbol representing the unknown number(s). The use of blanks also makes distinguishing between the blanks as being the same and/or different values is problematic. For instance, is the equation __ + __ = 12 equivalent to $x + x = 12$ or $x + y = 12$?

While elementary grade students are capable of making the transition from informal and idiosyncratic representations of unknowns to conventional algebraic notation for variables, they must be provided with experiences that promote and encourage meaning and sense making.

Unfortunately, the types of tasks in which students often engage in elementary grades typically include various representations with single value solutions. Over reliance on such tasks may contribute to common student difficulties and misconceptions related to variables (Knuth et al., 2005; Küchemann, 1981; McNeil et al., 2010) and promote students' fragile understanding of variable (Asquith, Stephens, Knuth, & Alibali, 2007; Marum et al., 2011)

**Students' Meanings for Variables**

Grade 4-6 participants solved tasks written in four different formats; equations with unknowns represented with shapes, blanks, or letters and a word problem (see figure 2). Responses were analyzed to determine if participants 1) distinguished between representations of unknowns and 2) the types of numbers they substituted for representations of unknown addends.

| Same addends | Different addends |
| --- | --- |
| • $y + y = 12$<br>• $\square + \square = 12$<br>• Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears. How many gummy bears could Shakira have? How many gummy bears could Tim have? | • $a + b = 12$<br>• $\_ + \_ = 12$<br>• $\square + \Delta = 12$<br>• Together Tom and Anne have 12 feet of ribbon. How long could Tom's ribbon be? How long could Anne's ribbon be? |

**Figure 2: Example Tasks**

**Distinguishing Between Variables**

Participants consistently treated the unknown addends in each equation as representing different quantities, such as different letters would be interpreted algebraically, regardless of whether the representations were the same (e.g., $y + y = 12$) or different (e.g., $\square + \Delta = 12$). Students gave solutions such as 10 and 2, 9 and 3, and 6 and 6 for each equation. While, the algebraic solution to $y + y = 12$ would be $y = 6$, participants did not believe that the different representations mattered. For example, when comparing in $a + b = 12$ and $y + y = 12$, participants stated that the only difference was that the letters were different. When asked if having two $y$'s or an $a$ and a $b$ made any difference, nearly every student said the equations meant the same thing and the different letters did not matter (Switzer, 2016).

Participants who did differentiate between the same and different representations of the variables, such as Sally in the opening vignette, did so inconsistently across equations with letters or shapes as the variables. In other words, Sally did not *always* treat different unknowns as different values. Participants who said that the same representation in an equation had to be the same value did not extend this same meaning to blanks. For example, in the equation $\_ + \_ = 12$ these students provided solutions where the blanks where the same value (i.e., 6 and 6) as well as different (e.g., 5 and 7). In this way, they appear to have generalized a different meaning for blanks than they did for letters and shapes.

In contrast, when presented as words in word problems the students could draw on the context to differentiate between unknowns. For instance, in the problem "Shakira and Tim have the same number of gummy bears. Together they have 12 gummy bears." nearly all students stated that Shakira and Tim had six gummy bears each. In contrast, for the problem "Together Tom and Anne have 12 feet of ribbon." participants provided multiple lengths for each ribbon.

**Solutions**

Participants tended to provide only whole number solutions. For instance, when asked if the $a$ in $a + b = 12$ could be two and a half, Julie indicated that it could not.

**Interviewer:** Would two-and-a-half work for this one [pointed to the $a$ in $a + b = 12$].

**Julie:** Maybe.

**Interviewer:** What do you mean, maybe?

**Julie:** Like there could be, okay, probably not.

**Interviewer:** Why not? Tell me what you were thinking about.

**Julie:** A half is not a number and I thought numbers could get answers like; it might have to be twelve-and-a-half.

Julie's belief that fractions were not numbers or acceptable solutions may be due, in part, to her prior experiences in mathematics where the majority of problems only used whole numbers. In other words, she may believe that whole numbers are the expected solutions. However, her claim that fractions are not numbers is concerning.

All of the participants in the study had pervious experiences operating with integers. Yet, when asked if there were any numbers that one of the unknown addends could not be, participants commonly stated that the addend could not be greater than the sum because you cannot add a greater number and get a lesser number. Fujii and Stephens (2008) found that students often employ boundary values, such as addends cannot be greater than the sum, in limiting the unknown addends to being less than or equal to the sum.

## Implications for Instruction

Students entering middle school are at a critical juncture in their mathematical career; transitioning from what is often an arithmetic focus to an algebraic focus, from operating on known numbers to working with variables, and from working with specific cases to making generalizations. Participants in this study demonstrated generalizations for representations of unknown addends and number substitutions that will likely make this transition difficult. Participants' meanings for unknown addends depended, at times, on the particular representation used and/or the type of problem in which it was used. Findings from this study demonstrated that participants' meanings for the various representations of unknown addends differed in important ways from the algebraic conventions for variables. Based on these results and what we know about students' meaning for letters as variables I provide three recommendations for instruction.

First, students need opportunities to engage in learning activities that include quantities that can vary, not just placeholders with a single solution. The participants in this study were able to draw on their number and operation sense to engage in simple addition tasks with unknown addends. In doing so, they provided evidence of their understanding of number, operation, and pre-understanding of the concept of variable, which are invaluable in selecting tasks and promoting productive mathematical discourse (National Council of Teachers of Mathematics, 2014) about interpreting and working with variables.

Second, while equations with a single unknown representing a single value are common in elementary grades, students need opportunities to explore tasks with more than one unknown. Using and connecting a variety of representations (National Council of Teachers of Mathematics, 2014) for equivalent tasks (e.g., $\square + \Delta = 12$ and $x + y = 12$; or $\square + \square = 12$, $y + y = 12$, and __ + __ = 12) that students must solve and compare can assist students in developing an algebraic understanding of variables and connecting letters as variables to their prior experiences with informal representations of variables.

For example, participants interpreted the shapes in $\square + \Delta = 12$ as different variables (i.e., the unknown addends can be the same and different values). When asked to rewrite the equation so that the two shapes *had* to be the same value, participants stated that it was not possible or were unsure if it was possible. Participants had not had experiences where writing an equation to represent a context where the unknown addends *had* to be the same value was required. For example, having students write equations to represent the following contexts, then compare and contrast the two equations and corresponding contexts provides an opportunity to "facilitate discourse among students" by having them "explain and defend their approaches" (NCTM 2014, p. 35)

1. Tim and Anne have the same number of Gummy Bears. Together they have the same number of Gummy Bears as Caleb.
2. Together Brandon and Jillian have the same number of Gummy Bears as Trung.

Such tasks can bridge their prior knowledge to algebraic conventions and provide opportunities to recognize the *need* for distinguishing between representations of the variables as different and the same values. If students, generate equivalent equations for the two contexts (i.e., different unknown addends and an unknown sum) such as $a + b = c$, where $a$ and $b$ represent the number of Gummy Bears that Tim and Anne have, or Brandon and Jillian, and $c$ represents the number of Gummy Bears Caleb or Trung have, ask students what numbers could be substituted for each unknown. Next, have students determine if these number substitutions would work for each corresponding context, which would lead to a discussion of the *need* for representing unknown quantities that *have* to be the same value. Further, such a discussion would help develop the quality and quantity of connections between the contexts and their corresponding equations.

Finally, participants consistently limited their solutions to whole and natural numbers even though they had worked with both fractions and negative numbers. One way to assist students in moving away from their use of only whole numbers is to engage them in explicit discussion of what values do *and* do not satisfy equations and why. Such discussions provide students the opportunity to consider *all* values when evaluating an equation or expression, not just those in the solution, which is especially important as students move from operating on whole numbers to integers and rational numbers.

For instance, when asked what the $x$ could be in $x + x = 7$, participants listed the pairs (0, 7); (1, 6); (2, 5); (3, 4) and the reverse of each. When asked if the $x$'s could be the same, most stated that they could not because adding a number to itself results in an even number, generalizing a property of whole numbers and extending it to all numbers. If the domain consists solely of whole numbers, and applying algebraic conventions, there is no solution to $x + x = 7$. If the domain includes the rational numbers, the solution to $x + x = 7$ is $x = 3\frac{1}{2}$.

Therefore, as students proceed from primarily operating on whole numbers, to fractions, integers, irrational numbers, and eventually the set of Real numbers, revisiting previous tasks, such as the previous example of $x + x = 7$ provides students the opportunity to consider and reason through what number substitutions will satisfy the equation, not satisfy the equation, and justify their conclusions. In doing so, teachers have the opportunity to reveal and make public students' underlying assumptions that lead to common number and operation misconceptions.

Karp, Bush, and Dougherty (2014) identified "rules that seem to hold true at the moment, given the content the student is learning" but eventually are not always true (p. 20). For example, one of the rules is, "you cannot take a bigger number from a smaller number" (p. 21). When students are only aware of, or only working with nonnegative integers this rule is "true". For example, no nonnegative integer solution exists for 5 – 7 = x. However, when the domain is extended to include negative integers, the rule "expires" since -2 is now included as a possible number substitution.

Addressing how rational numbers and integers can result in different number substitutions, or solutions sets, for the previously used equation has the potential to assist students in understanding the importance of what values, or number sets, to consider when solving an equation and lay the foundation for the concept of domain and range in algebra. Likewise, having students identify values that will not work for each set of values has the potential to assist students in recognizing the importance of considering *all* values and determining which do and do not satisfy the equation (i.e., what numbers are included *and* excluded from the domain).

## Conclusion

Elementary grade students' experiences with unknowns often results in incorrect or incomplete generalizations. As they move from an arithmetic focus to a conventional algebraic focus for the meaning of variables, these generalizations may compound students' difficulties learning algebraic conventions for variables. Having a better understanding of the meanings they have for informal representations of variables as they enter middle school provides teachers with tools and instructional strategies to support students' adoption of algebraic conventions.

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# What is Mathematical Modeling?

*By Christy D. Graybeal, Hood College and Francine Johnson*

The term *modeling* is used in several ways in mathematics education. For example, we talk about a teacher modeling mathematical procedures for students to replicate or students using physical models to represent mathematical concepts. The Common Core State Standards for Mathematics (CCSSM), however, use the term *modeling* differently. In Standard for Mathematical Practice Four, the term modeling is used to describe how "mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (CCSSI, 2010, p. 7). In the CCSSM high school standard on modeling, modeling is defined as "the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions" (p. 72). Furthermore, the modeling cycle shown in Figure 1 illustrates the process students should go through when modeling a situation.

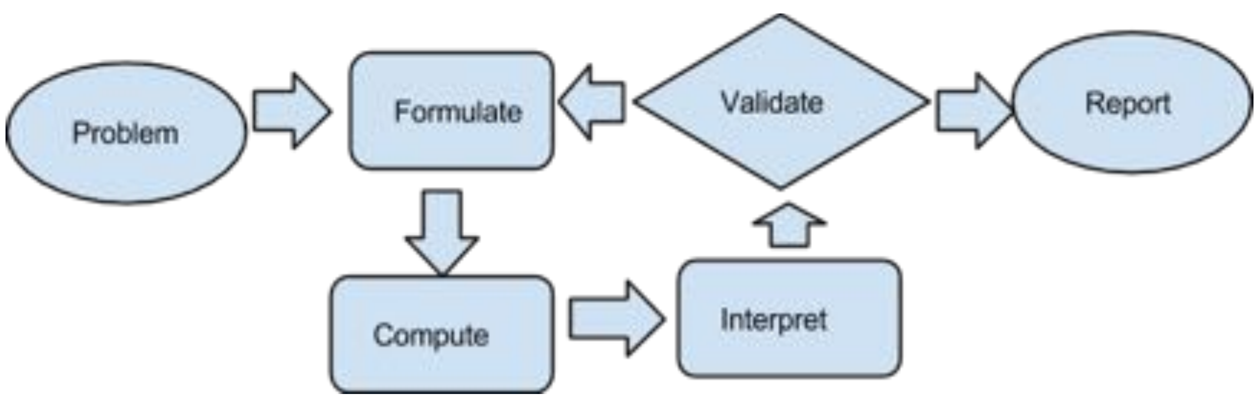


Figure 1. CCSSM modeling cycle (CCSSI , 2010, p. 72)

**Modeling versus Application**

Application tasks are often mistaken for modeling tasks. Ledder (2013) defines application problems as “having a narrow scope” with given numbers and conditions that call for “simply numbers” as solutions (p. 63). The focus is on the mechanics of mathematical calculations and their placement in a text usually indicates what computations are required. On the other hand, Ledder sees modeling as “mathematical constructions that describe real phenomena in the physical world” (p. 63). Modeling requires skills from many branches of mathematics and science and requires determining what best represents the real-world phenomena and will result in a meaningful solution. In short, modeling is messy.

In his TED talk, Dan Meyer illustrates how a task that initially only requires application of mathematical content can be transformed into a modeling task by stripping away much of the information given. In real life, students will rarely be told what question needs answered nor will they be given all of the information necessary to answer that question. We must get students in the habit of asking questions such as:

- What do we want to know?
- What do we know?
- Of this, what is important?
- What do I need to do with the important information?
- Does the result make sense?
- Do I need to try something else?
- Can I explain my result with supporting evidence?

**Sources of Modeling Tasks**

While most application problems in textbooks can be transformed into modeling tasks, there are several excellent sources of modeling tasks aligned to the secondary mathematics curriculum. These include:

- Consortium for Mathematics and Its Application. This website showcases all of the COMAP curriculum materials. Some materials are free and others can be purchased through the website.
- Dan Meyer's Three-Act Math Tasks. Dan Meyer's blog describes how three-act math tasks can be used in instruction and his spreadsheet of video tasks allows users to easily find modeling tasks relevant to almost any secondary mathematics standard. Free.
- Mathematical Modeling Handbook. This book contains 26 modeling modules developed to support teachers in implementing the CCSSM high school modeling standard. $45.
- Mathematical Modeling Handbook II: The Assessments. This supplement to the handbook provides examples of assessments for each of the modeling activities provided in the first handbook. $45.
- Model with Mathematics. This website is devoted to promoting the teaching and learning of mathematical modeling. Information ranges across the K-16 spectrum. Free.

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## Reader Interaction Opportunity

What do you consider to be a modeling task? Consider the following seven tasks. Put the tasks in order according to the amount of mathematical modeling students would engage in when answering the questions. (It is not necessary to answer the questions posed in the tasks.)

Submit your answers here:

https://www.surveymonkey.com/r/modelingmctm

In the next issue of the Banneker Banner we will present reader votes and compare them to results from a similar study.

## Modeling Tasks

**Telephone Charges** (Murdock, Kamischke, & Kamischke 2002, p. 210)
A long-distance telephone carrier charges \$1.38 for international calls of 1 minute or less and \$0.36 for each additional minute.
a. Write a recursive routine using calculator lists to find the cost of a 7-minute phone call.
b. Without graphing the sequence, give a verbal description of the graph showing the costs for calls that last whole numbers of minutes. Include in your description all the important values you need in order to draw the graph.

**Water Pressure on Diver** (Education Development Center 2009, p. 457)
When a diver goes under water, the weight of the water exerts pressure on the diver. The table shows how the water pressure on the diver increases as the diver's depth increases.
Water Pressure on a Diver

| Diver's Depth (ft) | Water Pressure (lb./in.$^2$) |
|---|---|
| 10 | 4.4 |

| | |
|---|---|
| 20 | 8.8 |
| 30 | 13.2 |
| 40 | 17.6 |
| 50 | 22.0 |

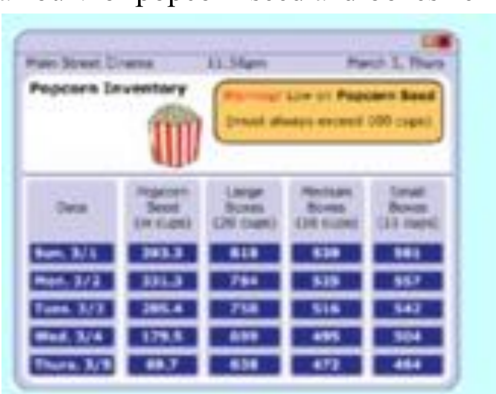
a. What is the water pressure on a diver at a depth of 60 feet? At a depth of 100 feet? Explain.
b. Write an equation describing the relationship between the depth $D$ and the pressure $P$.
c. Use your equation in part (b) to determine the depth of the diver, assuming the water pressure of the diver is 46.2 pounds per square inch. Explain.

**May's Hair** (Carter, Cuevas, Day, Malloy, Holliday, & Luchin 2010, p. 177)

May's hair was 8 inches long. In three months, it grew another inch at a steady rate. Assume that her hair growth continues at the same rate.
a. Make a table that shows May's hair length for each of the three months and for the next three months.
b. Draw a graph showing the relationship between May's hair length and time in months.
c. What is the slope of the graph? What does it represent?

**Popcorn Order** (PARCC 2013)

The Main Street Cinema gets a food delivery every Friday morning. On Thursday, Hannah checks the computer to determine what to order the next morning. The computer shows the amount of popcorn seed and boxes remaining at the end of each day.




Sales Sunday through Thursday are relatively consistent. Friday and Saturday are busier days, and on each of those days they sell between 200 and 300 large boxes of popcorn. On Friday and Saturday, they also sell about twice as many small and medium boxes of popcorn as they do on the other days.

She also knows that 1/3 cup of popcorn seed makes 8 cups of popcorn, and she must buy enough popcorn seed to last until the next delivery on the following Friday.

Estimate the amount of popcorn seed that Hannah should order this Friday so that there are between 100 and 200 cups of popcorn seed remaining next Friday morning. Show or explain the reasoning you used to determine your estimate.

**Predicted vs. Actual Calories** (COMAP 1998, pp. 362-363)

What is the relationship between the number of calories a food actually has and the number of calories people think it has? A food industry group surveyed 3368 people, asking them to guess the number of calories in several common foods.

| Food | Guessed calories | Actual calories |
|---|---|---|
| 8 oz. whole milk | 196 | 159 |
| 5 oz. spaghetti with tomato sauce | 394 | 163 |
| 5 oz. macaroni with cheese | 350 | 269 |
| One slice of wheat bread | 117 | 61 |
| One slice of white bread | 136 | 76 |
| 2-oz. candy bar | 364 | 260 |
| Saltine cracker | 74 | 12 |
| Medium-size apple | 107 | 80 |
| Medium-size potato | 160 | 88 |
| Cream-filled snack cake | 419 | 160 |

a) The goal is to predict the guessed calories from the actual calories. Enter the data into your calculator and make a scatter plot with this in mind.
b) Describe in words the most important features of the scatter plot.
c) Find the regression line for predicting guessed calories from actual calories. Then make a residual plot. Does the regression line adequately describe these data?
d) Would you classify any of the data as outliers? If so, identify them. What do they tell you?
e) If you found outliers, remove them and re-calculate the regression line. Compare your new equation to the one from part (c).
f) Do the actual calories in a food item enable you to predict accurately what people will guess? Explain.
g) Interpret the meaning of the slope of your model for predicting guessed calories from actual calories.

**Two-Second Rule** (Smarter Balanced 2014)

The "two-second rule" is used by a driver who wants to maintain a safe following distance at any speed. A driver must count two seconds from when the car in front of him or her passes a fixed point, such as a tree, until the driver passes the same fixed point. Drivers use this rule to determine the minimum distance to follow a car traveling at the same speed. A diagram representing this distance is shown.

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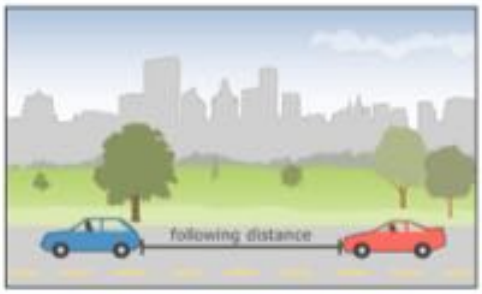


As the speed of the cars increases, the minimum following distance also increases. Explain how the "two-second rule" leads to a greater minimum following distance as the speed of the cars increases. As part of your explanation, include the minimum following distances, in feet, for cars traveling at 30 miles per hour and 60 miles per hour.

**Walking Trail** (PISA 2012)

The Gotemba walking trail up Mount Fuji is about 9 kilometres (km) long. Walkers need to return from the 18 km walk by 8 pm.

Toshi estimates that he can walk up the mountain at 1.5 kilometres per hour on average, and down at twice that speed. These speeds take into account meal breaks and rest times.

Using Toshi's estimated speeds, what is the latest time he can begin his walk so that he can return by 8 pm?

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# CALL FOR FEATURED MANUSCRIPT

The Banneker Banner

MCTM Maryland Council of Teachers of Mathematics

## FEATURED TOPIC: *Engaging Students in Mathematics*

***The Banneker Banner Editorial Panel continues to encourage classroom teachers, teacher educators, researchers, supervisors, and other practitioners to submit manuscripts that address the full range of topics of interest to teachers and students in PreK-12 mathematics. In addition to the Open Calls listed below, the Editorial Panel is inviting submissions that address the featured topic of Engaging Students in Mathematics.***

***According to the vision proposed by NCTM's Principles to Action, "effective mathematics teaching supports students in struggling productively as they learn mathematics. Such instruction embraces a view of students' struggles as opportunities for delving more deeply into understanding the mathematical structure of problems and relationships among mathematical ideas" (NCTM, 2014, p. 48). Featured articles will address the multiple ways in which teachers design and deliver instruction that afford students opportunities to engage in mathematics in a manner that supports their understanding of the relationships among mathematical ideas.***

***Engaging Students in Mathematics***

submissions due by January 15, 2017
publication MARCH 15, 2017

## OPEN CALL

**OPEN CALL manuscripts are focused on a particular theme or have a particular format. Open Call manuscripts do not have a submission deadline. It is a great way to begin sharing your ideas professionally. Themes for Open Call manuscripts include:**

*ACTIVITIES FOR STUDENTS* - Manuscripts describe student-centered activities suitable for immediate classroom use in grades Pre-K – 12. Authors share teacher implementation tips, differentiation options, and assessment strategies to promote discussion about important concepts, connections, and representations of the mathematical ideas central to a class or course.

*CONNECTING RESEARCH TO TEACHING* - Manuscripts describe lessons learned from research that informs classroom practice in a manner appropriate for teachers' application to classroom practice or otherwise suitable for reflective discussions at department meetings or other gatherings.

*LEARNING WITH TECHNOLOGY* - Manuscripts describe teachers' experiences and reflections on the meaningful use of technology to enhance instruction, assessment, or curriculum and how the use of technology has changed the teaching and learning of mathematics.

*MATHEMATICS TEACHING IN PreK - 12* - Manuscripts describe teachers' experiences bringing the mathematics curriculum to life in PreK to 12 classrooms. Teachers share their experiences and reflect on the successes and challenges of engaging young learners in the learning of mathematics.

*TEACHING MATHEMATICS TO DIVERSE LEARNERS*- Manuscripts describe teachers' experiences adapting mathematics curricula for diverse learners, such as Students with Disabilities, English Language Learners, and Gifted and Talented students. Teachers share their instructional practices and reflect on the successes and challenges of supporting diverse learners to reach their fullest potential as mathematics learners.

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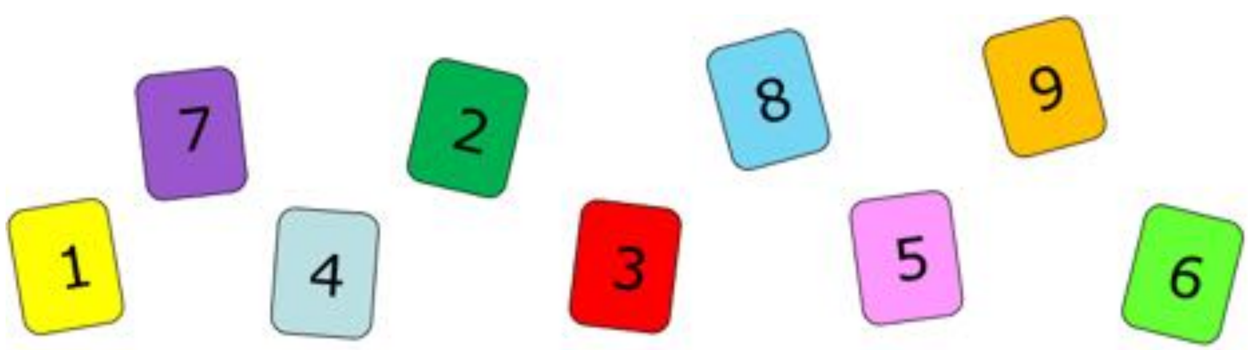
# Using Games to Promote Reasoning and Sense Making

*By Ann Holdren-Kong, National Council of Teachers of Mathematics*

Using games inside the classroom is most certainly a growing trend in education. However, amongst the plethora of resources available, it's not always apparent on what a good resource actually is. What's even more vexatious is how to extract a game's full learning potential. The games are only a vessel, and without a teacher, what a student learns is limited. The questions, pedagogy, extensions, assessments—*these* are the true resources that create learning. Such a vessel for making sense of math is The Game of Nine (or Sixteen) Cards, also known as Deep Sea Duel.

The game was originally known as, "What Is the Name of This Game?" by John Mahoney, which appeared in the October 2005 issue of Mathematics Teaching in the Middle School, vol. 11, no. 3, pp. 150–154. The title indicates that the game is an isomorphism of another game, which happens to be tic-tac-toe. The rules are simple. Two opponents take turns choosing one card at a time. The first one announce that they have a combination of three cards that add up to 15 wins.




Although the rules sound simple, it's not quite clear to students that one could end up with up to 5 cards. Here's an example of play. Let's assume that Jill is playing against Pete. Jill, going first, picks up the card 2. Then, Pete chooses 8, followed by Jill, who chooses 9. Next, Pete chooses 7. To recap, Jill has 2 and 9 (or a sum of 11), while Pete has 8 and 7 (a sum of 15). Note that at this point, Pete cannot claim a win, since he does not have a combination of **three** cards that sum to 15. So the play continues. Jill should pick up a 4, but let's say that she miscalculates and chooses a 3 instead. Now she has 3 cards that add up to 14 (2+9+3). The play's over, since neither person can get three cards that add up to 15, right? Wrong! They key word is *combination*. That means Jill can use her 2 and 9, 9 and 3, or 2 and 3 plus another card to get 15. But Pete's confused, so he randomly chooses a 6. Jill, understanding the rules, chooses a 4 next and declares her win with the cards: 2, 9, and 4.


So what's so special about this game? At first glance, we're just practicing addition with a bit of strategic play. But what if I told you that if Jill first chose an 8, and Pete chose a 3, that Jill can *guarantee* her win? Why is the sum 15? Does someone always win? Is there a best card? Now we're getting more interesting! These are the sorts of opportunities that technology alone cannot provide. Only a great teacher can find these questions and provide the guidance for true reasoning and sense making of mathematics.

So what is the strategy behind The Game of 9 Cards? As stated before, this game is an isomorphism to tic-tac-toe. How? You play against one opponent. You need three selections to win (but can make more than 3 selections). There are nine potential moves in each game. You can block an opponent. Each selection is unique. The list goes on, but the most important fact is that the playing board is the same— the numbers 1–9 can actually be imposed on a tic-tac-toe board such that each row, column, and diagonal will add up to 15. In mathematics, this is called a 3x3 magic square.

| | | |
|---|---|---|
| 8 | 1 | 6 |
| 3 | 5 | 7 |
| 4 | 9 | 2 |

Let's look at the pedagogy used to create this magic square. What are the right questions to ask students so that they are the creators of this magic square? After multiple plays, most students don't have a difficult time seeing that this game is indeed very similar to tic-tac-toe. Once they come to this conclusion, we can start listing all the combinations of 3 numbers that add up to 15. There are 8 such combinations. Each number is used 3 times, except for 5. The number 5 is actually used in 4 combinations. So it makes sense for students to put 5 in the center, as it is the only location where three-in-a-row can be achieved four times. The numbers 2, 4, 6, and 8 (the even numbers) are used three times, so it makes sense for those numbers to be in the corners of the magic square, since those are the location where you can get three-in-a-row three times. Note that the 8 and the 2 need to be in the same diagonal, since $8 + 5 + 2 = 15$. Similarly, 4 and 6 need to be in the same diagonal. The rest of the magic square is a matter of simple algebra.

Now, we have reduced the card game to a tic-tac-toe board. Having this by your side while playing the Game of Nine Cards reveals the strategy. So let's go back to an earlier question- If Jill first chose an 8, and Pete chose a 3, that Jill can *guarantee* her win? Play out a couple scenarios by yourself (or with a partner) on the board itself, using X's and O's on top of the numbers. Does the strategy reveal itself? See if you can answer all the questions that were previously asked.

Now the beauty of this game extends itself far beyond the magic square. Perform a linear transformation on the nine cards for an unlimited number of possible plays. For example, multiply all the nine numbers by 1.9 and then subtract them by 6. Now, yet another question arises. What should the sum be? Obviously, not 15. But how did we get 15 in the first place? See if you can study the nine numbers to find out.

And once you've figured that out, play with the numbers 1–16 and try to get a combination of 34. Is there a no-lose strategy in this game? Can a 4x4 magic square be created? Go and find out! As was explored, there were so many components to the Game of Nine Cards that would have been lost without deeply exploring the strategies and game play.

**Free Resources**

Free additional resources for Deep Sea Duel, along with downloadable worksheets, can be found at http://illuminations.nctm.org/deepseaduellp/. You can also get the free downloadable app at the iTunes or GooglePlay store. And for the web, a simple Deep Sea Duel search should get you there.

Reference

Mahoney, J. (2005). What Is the Name of This Game? Mathematics Teaching in the Middle School, 11(3), 150-154.

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# Using Transformations of Exponential Functions to Catch a Cold Blooded Killer

*By David S. Thompson, Baltimore City Public Schools*

Many times in mathematics classes we are left with the question, “When will I use this?” This is the question that intrigued me and drove me towards mathematics education—mathematical applications. One of the most exciting and intriguing activities I have written for students involves the use of exponential equations and Newton’s Law of Cooling. I remember as a mathematics major at Towson University, many of my colleagues found they were not as good with physics as they were with pure mathematics. As a project in my mathematics methods course in the fall of 2009, I developed an activity in which students could practice the mathematics they learn in Algebra II or a Pre-Calculus course with Newton’s Law of Cooling.

The idea of the activity was based on crime themed shows, which many students whom I observed during my student teaching enjoyed watching. In particular, students enjoyed the TV show *Numb3rs* and *CSI*. There was a college mathematics professor, Charlie Eppes who models the crimes with mathematics to give details to the FBI to help them solve the crime. While there wasn’t a specific episode where Eppes solved a murder using Newton’s Law of Cooling, Eppes probably would have used an exponential model or differential equations model in order to find the time of death if there were such an episode, because just like a cake cooling when it comes out the oven when a body dies, it cools to room temperature. The activity I created, allows students to solve a murder mystery in *Numb3rs* fashion, using algebra and logic. The activity presented in this article has been presented at the Association of Maryland Mathematics Teachers Educators 1st Annual Early Career Teachers Conference in a session entitled “Chilling out after a Murder" and published as a Calculus version in the North Carolina Council of Teacher of Mathematics Journal, The Centroid, in the Spring 2016 issue (Thompson, 2016).

**Teacher Directions for the Activity**

The teacher should begin with a discussion of Newton. Many students may recognize the name from Newton's three Laws of Motion; however, Newton also did work with thermodynamics, hence the naming the Newton's Law of Cooling. Ironically, Newton did not write the law in equation form as we are accustomed to seeing it in an algebra-based physics textbook. He stated the law as follows: "the excess of the degrees of the heat…were in geometrical progression when the times are in an arithmetic progression" (Newton, 1701). In other words, Newton noticed that as an object cooled, its temperature decreased exponentially as it approached room temperature. It wasn't until much later that the law was written in equation form as $T(t) = T_s+(T_0-T_s)e^{-kt}$, where $T_s$ = the temperature of the surroundings, $T_0$ = the initial temperature of the body, k = a cooling constant and $t = t_2-t_1$ (the lapsed time). In the Common Core Algebra 2 classroom, students should be able to differentiate between exponential growth and decay, this would lead to a good discussion of whether a body decreasing to the temperature of a room after its time of death would be an exponential growth or decay model. On the contrary, the teacher could pose the question what about an ice cube melting and its temperature as it approaches room temperature, would this be an exponential growth or decay model.

After the discussion, the teacher should distribute the activity sheets. To make the activity more interesting, the teacher could change the names of the people in the activity to names of students in his or her class. In fact, if the teacher changed the order of the names in the time card, with the exception of Jeff, the teacher could easily create a new crime scene with a different culprit. However, in the activity sheet the teacher should take note of the position of the four original suspects and their position on the time sheet and change them with their new alignment when they scramble the names. For roleplaying purposes this activity may be more effective as a partner's activity where students could play the role of police partners and the teacher can play the role of police chief. Teachers may need to remind students the relationship of base *e* and the natural logarithmic function *ln*, as students are manipulating the initial equation to find the cooling constant. To make sure students arrive at the correct equation, the teacher may want to help students derive this literal equation. After students have solved the literal equation, the teacher may need to remind students that to solve for a value should substitute the values in and solve, which will give them the cooling constant. Once students have solved for all these values, the teacher may need to remind them where to substitute the values in to find the time of death, again the key to solving for the time of death is proper substitution and the fact that exponential base *e* and the natural logarithm *ln* are inverses. A duplicate copy of the time card has been provided so teachers can use half a sheet of paper when distributing time cards.

In order to analyze the function and the situation in problems 8-11, the teacher should ask questions such as: For what values does the domain and range make sense for in this situation? How do I solve for time if I know the temperature? What given information can help us determine what the values *a, b, h,* and *k* mean in the real world situation. As an extension of this activity, teacher may want to have students write their own crime word problem for others to solve to find the time of death of the body.

| Common Core State Standards: Standards for Mathematical Practice | |
|---|---|
| MP1: Make sense of problems and persevere in solving them. | Students need to understand the entry point into the problem. Before students can proceed with finding the death of the casino manager, students must be able find the cooling constant. First, students must also be able to make sense of all the givens in the problem. Then, students must understand that exponential equations of base *e* and the natural logarithmic equations are inverses of each other in order to solve for the cooling constant. |
| MP2: Reason abstractly and quantitatively. | Students must be able to decontextualize the information given in the problem to be able put the givens into an abstract equation and solve the problem at hand. Students should also be able to make sense of the units involved in the problem, temperature and time. |
| MP4: Model with mathematics. | In this problem, students use the mathematics they learn through solving exponential and logarithmic equations to solve a real world problem, namely a time of death problem to solve a murder mystery. |
| MP5: Use appropriate tools strategically. | Students must be able to use a graphing calculator or scientific calculator in order to find the exact time of death when solving for the cooling constant. |
| MP6: Attend to precision. | After calculating the cooling constant, students must decide whether to round the answer or use the exact answer the calculator gives the student for the cooling constant. |
| **Common Core State Standards: Content Standards** | |
| SSE.A.1: Interpret expressions that represent a quantity in terms of its context. | Students must be able to interpret the expression $T_s + (T_0 - T_s)e^{-kt}$, in terms of the context of the problem, namely an expression relating temperature and time to solve for the current temperature of an object being affected by the temperature of its surroundings. |
| CED.A.1: Create equations and inequalities–in one variable and use them to solve problems. | Students must be able to take the givens in the problem and use them to create an equation to solve the problem using the equation $T(t) = T_s + (T_0 - T_s)e^{-kt}$. Students first must be able to solve for the ungiven cooling constant, and then use this information to solve for the time of death. |

| | |
|---|---|
| CED.A.4: Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. | Students must be able to rearrange the equation $T(t) = T_s + (T_0 - T_s)e^{-kt}$ to solve for k. Students must be able to use algebra methods to isolate and solve for k. Namely, students must know that exponential and logarithmic equations are inverses and subtraction and division are the inverse operations of addition and multiplication. |
| IF.A.2<br>Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. | Students must use function notation to complete the table and find the temperature at time $t$ of the Newton's Law of Cooling Function. $T(t) = T_s + (T_0 - T_s)e^{-kt}$. |
| BF.B.3<br>Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of $k$ (both positive and negative); find the value of $k$ given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. | Students should be able to explain the effects on the graph of the parent exponential function that the quantities *a, h,* and *k* have. Students should be able to interpret this from the graph and using the equation of the transformed function. |
| F.B.5<br>Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. | Students must find the domain of the function in terms of the function and also interpret the situation to determine the domain of the function situation that this function describes. |
| LE.A.1.C<br>Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. | Students should be able to recognize from the description, the graph, and the table that Newton's Law of Cooling represents an exponential function. Furthermore since the graph and table show decreasing numbers, student's should be able to determine that Newton's law of Cooling represents exponential decay. |
| LE.B.5<br>Interpret the parameters in a linear or exponential function in terms of a context. | Students should be able to interpret the various parts of the parent exponential function $T(t) = a \cdot b^{-kt-h} + k$, in terms of he context of the problem. |

***Table 1:*** **Common Core State Standards for Mathematics addressed in this activity.**

## CSI: Murder at Hollywood Casino

**Directions:** You and your partner are students at the police academy and have been requested to help the State Police Crime Scene Investigation Unit. You are investigating the murder of a Hollywood Casino manager; however, there is little evidence available to solve this murder mystery as the murder weapon has disappeared. You must figure out the time of death to determine a time frame of the murder. Upon determining the time frame, you and your partner will request a warrant to view the time card report of the restaurant for the day the murder occurred. You will use this information to determine who is the prime suspect.

**Known Facts:**

1. There are four sets of finger prints on the manager's desk, where the manager was found dead.
    a. The first set has been discovered to belong to the manager, Jeff.
    b. The second set has been discovered to belong to Sean, the cook.
    c. The third set has been discovered to belong to Victoria, the waitress.
    d. The fourth set has been discovered to belong to Heather, the host.
2. The temperature of the restaurant remains at a constant 70°F.
3. The temperature of the body at the time of death was 98.6°F. (Assuming the victim was not sick at the time of death.)
4. The temperature of the body at 7:00 am was recorded as 80.1°F.
5. The temperature of the body twelve hours later was 71.1°F.

**Question:** What important pieces of information are needed to begin zeroing in on a suspect?

**Newton's Cooling Law**

Based on experimental observations, it has been proven that the temperature of an object changes at a proportional rate to the temperature of its surroundings. This relationship can be given by the equation $T(t) = T_s + (T_0 - T_s)e^{-kt}$.

$T_s$ = the temperature of the surroundings

$T_0$ = the initial temperature of the body

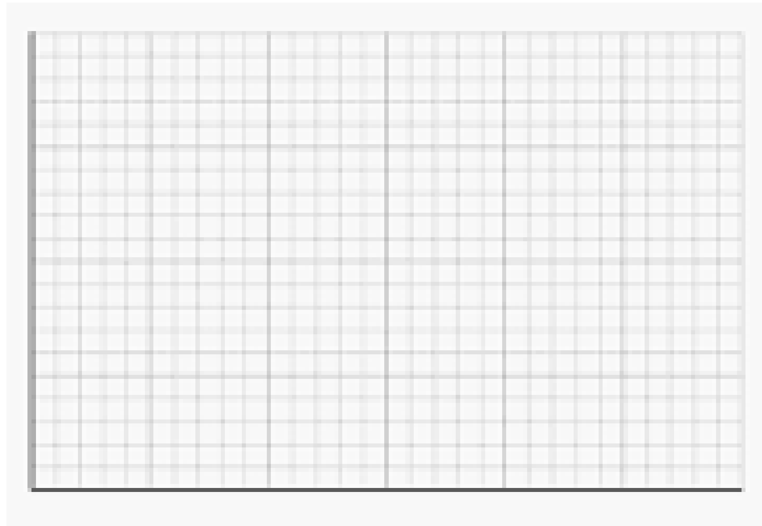
K = a cooling constant

$t = t_2 - t_1$ (the lapsed time)

1. Manipulate the equation to determine an equation to calculate k, the cooling constant. (Hint: The inverse of the natural number $e$ is the natural logarithm ln(x).)
2. Substitute your known values to calculate k, the cooling constant. (Hint: T(t) = T(12) = 71.1°F and $T_0$ = 80.1°F.)
3. Use the cooling constant you calculated in step 2 to calculate how much time has elapsed since the victim was murdered. (Hint: $T_0$ = 98.6°F T(t) = $T_0$ from part 2.)
4. Ask for a warrant to view the time card report.
5. Based on the time card report and your calculations on the time of death who is the prime suspect? Explain.
6. Explain how the time of death can be thrown off if the body was in a freezer at 32°F for three hours after the time of death, prior to being placed at the desk. Would you expect the body temperature to decrease at the same rate in the freezer as it would in a room at 70°F? (Hint: Solve $T(3) = T_s + (T_0 - T_s)e^{-kt}$ for $T_s$ = 65°F and $T_s$ = 32°F.)

7. Complete the following table. Round to 3 decimal places. Using the graph sketch a graph of this situation.

| Time (t) | Body Temperature T(t) |
|---|---|
| 0 | |
| 1 | |
| | 89.764 |
| 3 | |
| | 83.658 |
| | 70.001 |



8. Using the graph, you sketched in problem 7 find the domain, range and any asymptotes for the function and the situation. Also determine whether the graph is a model of exponential growth or decay. Explain.
9. Using the table from number 7 write an exponential function in the form $T(t) = a \cdot b^{x-h} + k$.
10. Define the base and describe the transformations based on the equation that you found in number 8.
11. Explain what the values of $a$, $b$, $h$, and $k$ represent in the real world situation.

| Time Card Report for 06/04/2011 | | | | |
|---|---|---|---|---|
| **Name** | **Date** | **Scheduled Shift** | **Actual Hours** | **Total Hours Worked** |
| Summer | 10/31/2009 | 10:30am-6:30pm | 10:30am-7:00pm | 8.5 hours |
| Mycaylah | 10/31/2009 | 10:30am-7:00pm | 10:30am-7:15pm | 8.25 hrs |
| Victoria | 10/31/2009 | 10:30am-6:30pm | 10:30am-7:00pm | 8.5 hrs |
| Nathan | 10/31/2009 | 11:30am-7:30pm | 11:30am-8:00pm | 8.5 hrs |
| Kayla | 10/31/2009 | 10:30am-8:00pm | 10:00:00 AM-8:15pm | 9.75 hrs |
| **Jeff** | 10/31/2009 | **4:00pm-1:15am** | **4:00pm-** | **Not Clocked Out** |
| Nehemiah | 10/31/2009 | 5:00pm-11:00pm | 5:00pm-11:30pm | 6.5 hrs |
| Sean | 10/31/2009 | 4:00pm-9:00pm | 4:00pm-9:00pm | 5 hrs |
| Emily | 10/31/2009 | 4:00pm-10:00pm | 4:00pm-10:30pm | 6.5 hrs |
| Raven | 10/31/2009 | 3:00pm-10:00pm | 3:00pm-12:00am | 7.5 hrs |
| Molly | 10/31/2009 | 3:00pm-11:00pm | 3:00pm-12:00am | 9 hrs |
| Ryan | 10/31/2009 | 5:00pm-12:30am | 5:00pm-12:30am | 7.5 hrs |
| Heather | 10/31/2009 | 5:00pm-1:15am | 5:00pm-1:15 am | 8.25 hrs |


| Time Card Report for 06/04/2011 | | | | |
|---|---|---|---|---|
| **Name** | **Date** | **Scheduled Shift** | **Actual Hours** | **Total Hours Worked** |
| Summer | 10/31/2009 | 10:30am-6:30pm | 10:30am-7:00pm | 8.5 hours |
| Mycaylah | 10/31/2009 | 10:30am-7:00pm | 10:30am-7:15pm | 8.25 hrs |
| Victoria | 10/31/2009 | 10:30am-6:30pm | 10:30am-7:00pm | 8.5 hrs |
| Nathan | 10/31/2009 | 11:30am-7:30pm | 11:30am-8:00pm | 8.5 hrs |
| Kayla | 10/31/2009 | 10:30am-8:00pm | 10:00:00 AM-8:15pm | 9.75 hrs |
| **Jeff** | 10/31/2009 | **4:00pm-1:15am** | **4:00pm-** | **Not Clocked Out** |
| Nehemiah | 10/31/2009 | 5:00pm-11:00pm | 5:00pm-11:30pm | 6.5 hrs |
| Sean | 10/31/2009 | 4:00pm-9:00pm | 4:00pm-9:00pm | 5 hrs |
| Emily | 10/31/2009 | 4:00pm-10:00pm | 4:00pm-10:30pm | 6.5 hrs |
| Raven | 10/31/2009 | 3:00pm-10:00pm | 3:00pm-12:00am | 7.5 hrs |
| Molly | 10/31/2009 | 3:00pm-11:00pm | 3:00pm-12:00am | 9 hrs |
| Ryan | 10/31/2009 | 5:00pm-12:30am | 5:00pm-12:30am | 7.5 hrs |
| Heather | 10/31/2009 | 5:00pm-1:15am | 5:00pm-1:15 am | 8.25 hrs |

**CSI: Murder at Hollywood Casino—Answer Key**

**Question:** What important pieces of information are needed to begin zoning in on a suspect? **The time of death, who was nearby at the time of the murder, DNA evidence if available, finger prints, etc.**

1. Manipulate the equation to determine an equation to calculate k, the cooling constant.

   (Hint: The inverse of the natural number *e* is the natural logarithm ln(x).) $\frac{\ln\left(\frac{T(t)-T_s}{T_0-T_s}\right)}{-t} = k$

2. Substitute your known values to calculate k, the cooling constant.

   (Hint: T(t)= T(12)=71.1°F and $T_0$=80.1°F.) $\frac{\ln\left(\frac{71.1-70}{80.1-70}\right)}{-12} \approx .184769$

3. Use the cooling constant you calculated in step 2 to calculate how much time has elapsed since the victim was murdered. (Hint: $T_0$ = 98.6°F T(t) = $T_0$ from part 2.)

4. Ask for a warrant to view the time card report.

5. Based on the time card report and your calculations on the time of death who is the prime suspect? Explain.

   **Heather, her time card punches are closest to the time of death based on the calculations.**

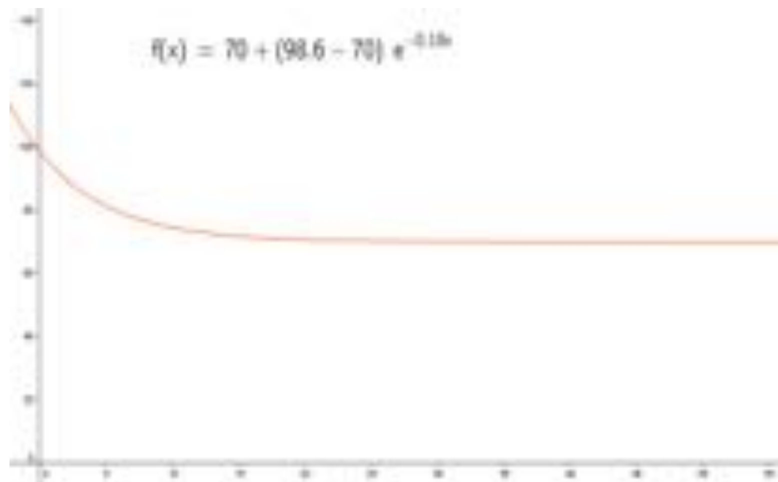
6. Explain how the time of death can be thrown off if the body was in a freezer at 32°F for three hours after the time of death, prior to being placed at the desk. Would you expect the body temperature to decrease at the same rate in the freezer as it would in a room at 70°F?

   **No, when you solve for the cooling constant, k, the temperature of the surroundings would affect the cooling constant. Therefore, the temperature of the body should**

**decrease at a more rapid rate and if it was placed back at the desk it would make it seem like the manager was dead longer than he actually was.**

7. Complete the following table. Round to 3 decimal places.

| Time (t) | Body Temperature T(t) |
|---|---|
| 0 | **0** |
| 1 | 93.775 |
| **2** | 89.764 |
| 3 | **86.43** |
| 4 | 83.658 |
| **55.535** | 70.001 |




8. Using the graph, you sketched in problem 7 find the domain, range and any asymptotes for the function and the situation.

**The domain of the graph for the function is $(-\infty,\ \infty)$, and for the situation would be $[0,\ \infty)$ (students could argue that negative time is before the time of death; however, then human body temperatures would be much greater than 98.6° which is not true). The range of the graph for the function is $[70,\ \infty)$, and for the situation would be $[70,\ 98.6]$. There is a horizontal asymptote for the function and the situation at $y = 70$. This is an exponential decay model since the graph is decreasing exponentially as time increases.**

9. Using the table from number 7 write an exponential function in the form $T(t) = a \cdot b^{x-h} + k$. => $\boldsymbol{T(t) = 98.6 \cdot (.96)^{-x} + 70}$

10. Define the base and describe the transformations based on the equation that you found in number 8.

**The base is .96, the graph is being stretched vertically by a factor of 98.6, shifted up**

**vertically by 70, and the negative in front of x reverses the position along the x-axis where the exponent is negative.**

11. Explain what the values of $a$, $b$, $h$, and $k$ represent in the real world situation.

**For $a = 98.6$, this represents the starting temperature of the human body before it begins losing temperature after the murder. For $b = .96$, this represents the average percent of temperature decrease per hour. For $k = 70$, this represents the room temperature.**

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# Are Bowling Pins the New Dice?

*By Matthew Wells, Montgomery County Public Schools*

A set of plastic bowling pins, a foam ball, and some creativity can serve as an upgrade to dice, with more flexibility, options, and most importantly, increased student engagement. Bowling pins can be adapted in many ways to turn typical matching and number generating into fun and memorable experiences for individual students or small groups. I have used them in my math intervention class for grades 2-5, but their use can be further widespread. Bowling pins promote equity, social/emotional learning, student discourse, and engagement. At first glance, activities like this may seem basic, but as students participate they become motivated and utilize higher order thinking skills.

Bowling pins promote equity by giving students an alternative way to communicate. I work with a student who is selectively mute, and giving her an avenue to participate without speaking, allows her to shine. Further, her anxiety level drops, as the environment becomes more accepting to a student under these unique circumstances.

By participating in activities with the bowling pins, social and emotional growth is paired with learning math. Students are exposed to games that involve working collaboratively, serving as a leader, winning and losing. Student discourse is necessary as students play and solve the problems that are presented.

More specifically, these activities support several of the Social and Emotional Learning Competencies defined by the Collaborative for Academic, Social, and Emotional Learning (CASEL). Self-awareness increases as students feel an increased sense of confidence and optimism in math. Self-management will be strengthened as students learn to regulate emotions (winning and losing), control impulses, and motivate themselves to progress further. Lastly, relationship skills will advance as these activities promote the ability to communicate clearly, listen actively, and cooperate with one another. As outlined by CASEL, one approach to Social and Emotional learning is integration with academic curriculum areas, and the use of the bowling pins in the following examples provides a clear way to implement this integration. By planning for, and teaching lessons that utilize social and emotional learning, a teacher is not only imparting knowledge, but equally as important, engaging their students in the CASEL outcomes; positive social behavior, fewer conduct problems, less emotional distress, and academic success (CASEL, 2012).

As with the implementation of any new classroom activity, it is important to set clear expectations prior to beginning. In this case, we thoroughly reviewed not only the directions of each of the games and the math involved behind them, but also the physical expectations of playing with bowling pins in class. I modeled how to roll the ball, and then did a think aloud of how to solve various problems.

**Examples of activities that I have integrated into class include:**

Multiplication Practice (Common Core State Standard Alignment: 3.OA.C.7)

1. Set up a standard triangular bowling formation with ten pins.

2. Have each student roll the ball twice per turn. Each pin knocked over represents a focus number. Students create products by multiplying their two factors (the focus number and the number of pins they knock down). If six is the focus number and a student knocks down five pins, their score would be 30 (6 x 5). If, on their next roll, they knock over two more, that is 12 more points, 30 + 12, 7 x 6, or 42.

3. Encourage students to use flexibility when solving their problems. The roll mentioned above could be solved in a number of ways: (6 x 5) + (6 x 2), 6 x 7, 30 + 12, (6 + 6 + 6 + 6 + 6) + (6 + 6), etc. Students will explain their answer to their peers.

4. The highest score wins the round. After each round, change the focus number.

Higher Order Thinking Questions: How many pins would you need to knock down to reach another student's score? What is the most efficient strategy to get your answer? Which is worth more, 6 five-point pins, or 4 eight-point pins?

Fraction Practice (CCSS: 4.NF.B.3, 5.NF.B.4)

1. Set up a standard triangular bowling formation with ten pins.

2. Have each student roll the ball twice per turn. Each pin knocked over represents a given focus unit fraction.

3. As students knock down pins, they practice composing fractions, when appropriate, naming them as a fraction greater than one and a mixed number, and then adding them together. For example, if the given focus fraction is 1/3, and the student knocks over two pins, and then three pins, they would first compose 2/3 and 3/3 and then add them to get 5/3 or 1 2/3. On each of their rolls, students will view their task as both repeated addition of the given unit fraction and multiplication of the given unit fraction by a whole number (the number of pins they knocked down). After their two rolls, students will add the two fractions or mixed numbers with like denominators that they composed on each roll.

4. After each round, change the focus fraction.

Higher Order Thinking Questions:

What two whole numbers is your fraction between?

Which whole number is it closest to?

What would the value of a strike (ten pins knocked down) be?

Which student has the largest/smallest fraction?

How many pins would you need to knock down to get a given fraction?

How could you show your solution by multiplying a unit fraction by a whole number?

How many pins would you have to knock down to get a whole number?

Addition Practice (CCSS: 1.OA.B.3, 2.OA.B.2)

1. Write a single digit number on the bottom of each pin with a dry erase marker.

2. Set up a standard triangular bowling formation with ten pins.

3. Have each student roll twice each turn.

4. Students will add up the number on the bottom of the pins for each roll, and then add the two rolls together.

5. Encourage students to explain their process in solving the problems. What strategies did they use? (Making ten, double facts, etc.)

Higher Order Thinking Questions:

Would you have used the same strategy as this student?

Can you solve it in a different way?

How much larger is a student's score from another student's score?

What number would have to be under this pin for your score to be ___?

Matching Products/Sums (CCSS: 3.OA.C.7, 2.OA.B.2)

1. Create ten multiplication or addition facts on index cards.

2. Write corresponding products/sums on the front of the pins with a dry erase marker.

3. Line up the pins with the answers visible in a horizontal line.

3. Teacher (or student leader of the game) holds up a card for a student, who finds the answer, and has three chances to knock down the pin with the match. If the pin is knocked down, that card and pin are eliminated from the game. If they do not knock down the pin, it can be asked again.

4. Teacher (or student leader) keeps presenting cards to participants until there are no more pins and cards.

Higher Order Thinking Questions:

Is the game easier towards the beginning or the end, and why?

How did you solve your problem?

Can you solve it a different way?

What expression would you need to knock down this pin?

Each of these games can be differentiated by level by changing the numbers involved. I also change some of the procedures for students of different ages, allowing older students to lead themselves and providing more support for younger students. Very quickly, my students both looked forward to and became accustomed to these activities. My upfront time investment paid dividends as the room was buzzing with math. Students began to organically make generalizations about numbers and pick up on patterns. The structure of these activities encouraged a higher level of reasoning. For example, when practicing multiplication, students need to not only know the answer, but also how many more pins they need to knock down to get the highest score. I discovered that with the bowling pins, students independently made connections and leaps in learning that are usually teacher-assisted.

After some reflection, what became evident is that these pins are simply a vehicle to help the math come alive for kids. I designed the aforementioned activities for specific kids and topics, but the scope of their use is very wide. When speaking to my students about the bowling activities, they volunteered, “It’s like bowling, but better,” “we did math and had fun at the same time,” and “I like it because you have to use your strength.” I’m not sure if that student even realized it, but he was using his math strength.

Bringing bowling pins, something students like outside of school, into the classroom yielded increases in engagement, motivation and achievement. My goal was to provide students with an active learning experience to practice and solidify learned skills. You might want to try it with your students - it could be right up their alley!

References

Collaborative for Academic, Social, and Emotional Learning. (2012, September). *2013 CASEL Guide: Effective Social and Emotional Learning Programs-Preschool and Elementary School Edition.* Retrieved from www.casel.org.

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For Maryland teachers evaluated with Danielson, attending a professional conference could be an artifact for Components 4d: Participating the Professional Community and 4e: Growing and Developing Professionally.

**Conference Program Information:**
The 2016 Annual Conference program will be available online in mid-September.
**Conference Registration Information:**

Advanced Registration is available online at
https://www.marylandmathematics.org/store/c1/Featured_Products.html
Onsite registration will be available on the day of the conference.

***We hope you can join us!***

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**The Maryland Council of Teachers of Mathematics (MCTM) is the professional organization for Maryland's teachers of mathematics. Our members represent all levels of mathematics educators, from preschool through college. We are an affiliate of the National Council of Teachers of Mathematics (NCTM). Our goal is to support teachers in their professional endeavors and help them to become agents of change in mathematics education.**

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**Join on-line at https://www.marylandmathematics.org/ or mail a $15 check, made out to MCTM, and this completed form to:**

**Mrs. Holly Cheung**
**Treasure**
**Howard County Public School System**
**10910 Clarksville Pike**
**Ellicott City, MD 21042**

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Furthermore, the MCTM Board invites all members to become actively involved in our organization. To become involved, please contact one of the officers listed above. We would love to hear from you!

**Maryland Council of Teachers of Mathematics**
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**MCTM Mission Statement:** The MCTM is a public voice of mathematics education, inspiring vision, providing leadership, offering professional development, and supporting equitable mathematics learning of the highest quality for all students.