Lesson Seed

Fractions for All: Instructional Task Sequences in a “Teaching Without Telling” Approach

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Abstract

This article introduces instructional task sequences aimed at enhancing conceptual understanding of fractions through a "teaching without telling" approach. It emphasizes mental activities such as partitioning and iterating within a measurement context, focusing on the fraction-as-a-measure meaning. Utilizing Cuisenaire Rods and virtual manipulatives from the Mathsbot platform, the task sequences progress from physical to symbolic representations. These sequences aim to build foundational knowledge and support students in developing their own understanding of fraction concepts.

Introduction and Background

When teaching fraction concepts, do you often find yourself telling students the next steps in a solution process instead of allowing them to determine the solutions on their own? Have you ever wondered how effective your lessons are in reaching all students, and not just those who are already good at mathematics? Are you able to support your students struggling with fractions or those facing strong anxiety about them? Despite decades of work in mathematics education research, as well as its emphasis in the curriculum, students are still struggling to understand fraction concepts. Data from the National Assessment of Educational Progress (NAEP, n.d.) consistently provide evidence of students’ struggles with fractions. For example, according to the 2017 NAEP, only 32% of fourth graders correctly determined which fractions were greater than, less than, or equal to the benchmark fraction, 1/2. In the 2022 NAEP, only 26% of fourth graders accurately assessed the correctness of statements involving fraction comparisons. Fractions are one of the primary and foundational content domains of the K-6 curriculum. Yet, fraction instruction is often unsuccessful. The traditional show-and-tell instructional approach is not, nor should it ever have been, the solution. A new approach is needed to foster mathematical understanding for fractions.

In this article, I share instructional task sequences for fractions that are designed with the “teaching without telling” approach in a measurement context. The task sequences are informed by prior research work (Simon, Kara, Placa, & Avitzur, 2018). The premise of that research was that if a mathematical topic can be taught with conceptual understanding (e.g., being able to explain why a rule works), rather than procedural understanding (e.g., just being able to apply the rule), then it should be possible to engineer a sequence of tasks that promotes conceptual understanding without relying on the show-and-tell method. Using this lens, the task sequences shared in this article were designed within the “teaching without telling” approach. In the “teaching without telling” approach, students are provided with a sequence of tasks that begin by applying their current knowledge and build upon it by solely engaging with these tasks. Using their current knowledge, students solve the tasks and progress in noticing patterns that eventually enable them to explain the underlying logical necessities that create the rules and formulas, ultimately developing their conceptual understanding. The instructional task sequences shared in this article focus on mental activities, such as partitioning and iterating, by promoting fraction-as-a-measure meaning in a measurement context. In the following sections, I explain these terms and then discuss the instructional task sequences.

Fractions for All *(continued)*

Meanings of Fractions

Fractions have various meanings, including part-whole, measurement, division (quotient), operator, and ratio. Table 1 provides examples of these meanings in the context of 4/5. (Note: For formatting consistency, fractions are written in this form throughout the article. Students should be presented with the traditional fraction symbols). The Common Core State Standards for Mathematics (CCSS-M) illustrate the progression from the part-whole meaning in earlier grades, such as 2.G.A.3 and 3.NF.1, towards the fraction-as-a-measure meaning in later grades, such as 4.NF.4a (CSSI, 2010).

Table 1

Meanings of Fractions (adapted from Norton & Boyce, 2013)

Meaning	Example representations for 4/5
Part-whole	A whole is partitioned into five equal parts and four of those parts are shaded.
Measure	Four iterations of a length 1/5 that when iterated five times would give a length of one unit
Division	The amount each person would get when $4 were shared equally among five people
Operator	A rule for making four-fifths of some other amount: 4/5 of 35 square meter
Ratio	Four dollars per five cereal bars

In earlier U.S. textbooks, Alajmi (2012) found that only the part-whole meaning of fractions was emphasized. Although the part-whole meaning is necessary to build fraction concepts in early grades, it is not sufficient for advanced fraction topics (e.g., fractions greater than one) and does not help to build the understanding that "fractions are numbers" on the number line. On the other hand, the fraction-as-a-measure meaning focuses on the length of a bar, which facilitates a smooth transition to understanding fractions as numbers. This meaning has proven to be promising in promoting conceptual understanding (Lamon, 2007) and addressing students' difficulties in learning fractions (e.g., Simon, Placa, Avitzur, & Kara, 2018). For example, the representation of fractions that are greater than one, such as 6/5, is compatible with the fraction-as-a-measure meaning, as 6/5 is interpreted as six iterations of the partial unit, 1/5.

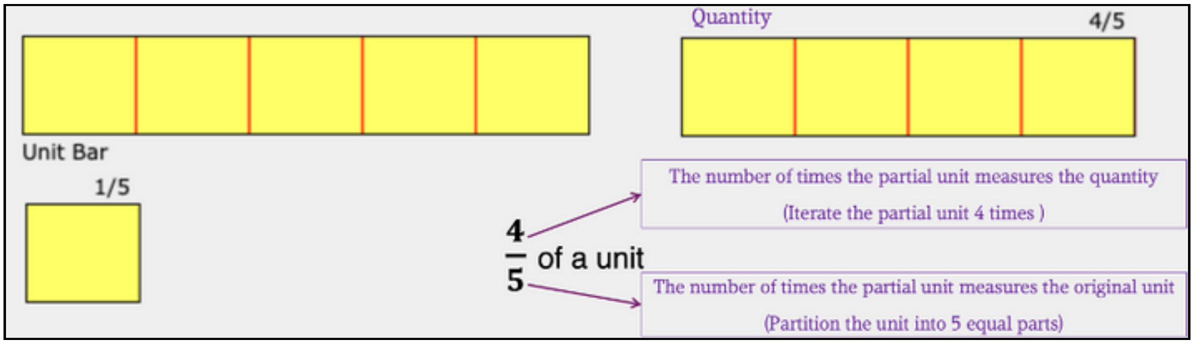
Measurement Approach

In this article, the fraction-as-a-measure meaning is explored in a particular measurement context, focusing on the question: What happens if the unit does not measure the quantity an integer number of times? (e.g., Simon, Placa, Avitzur, & Kara, 2018). It would mean that to be able to measure the quantity, we would need a partial unit, such as 1/b, that is created by partitioning the unit into b number of pieces. Next, we would measure the quantity by iterating the partial unit a times and identify the quantity as a/b of a unit long. (see Figure 1 for an example). Note that the partitioning and iterating actions, as well as the size and number of pieces resulting from these actions, are the mental actions involved in this approach.

Fractions for All *(continued)*

Figure 1

Description of a measurement context

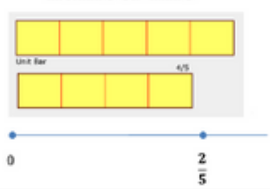


Fraction Models

Fractions can be represented in area, length, or set models (see Figure 2 for example representations). Depending on the meaning of the fraction, one model can be more beneficial than another. For the tasks discussed in this article, the length model is the focus.

Figure 2

Fraction Models

Area model	Length model	Set model
Drawing rectangles, pattern blocks 	Length representation or number line 	Discrete objects 

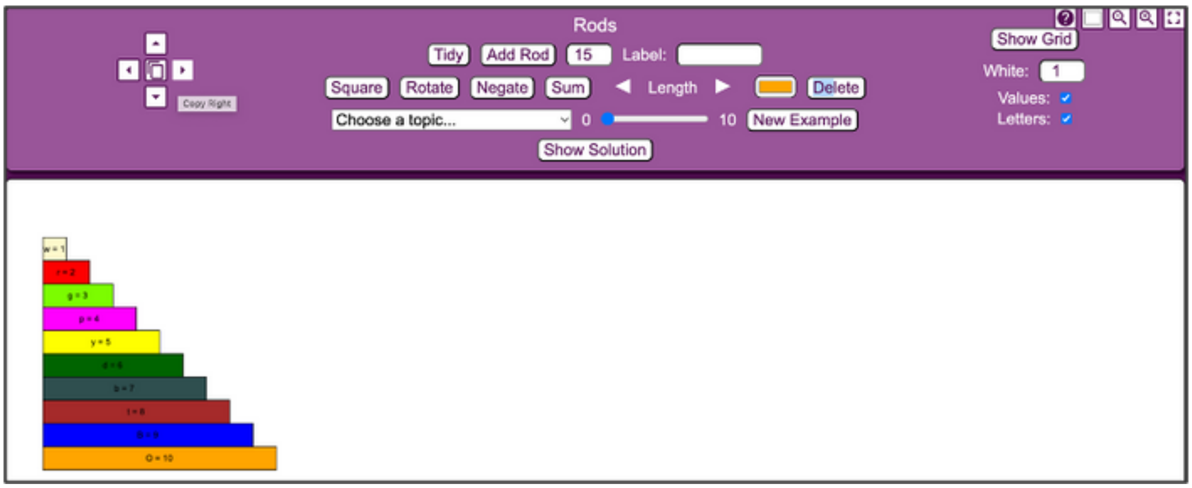
Representation/Manipulative Selection

The primary purpose of using manipulatives is to aid activities that might not be immediately accessible to students (either mentally or symbolically) but are foundational for understanding. Task sequences shared in this article will facilitate a gradual transition from the use of manipulatives towards symbolic representations. Because the tasks require both physical and mental actions, such as partitioning and iterating, the selected manipulatives should support these actions.

The tasks in this article focus on utilizing Cuisenaire Rods, which are three-dimensional or virtual bars with varying lengths. The virtual manipulative used in this article is “Rods” in Mathsbot (n.d.), a digital platform that provides a two-dimensional version of Cuisenaire Rods. Figure 3 includes one set of rods for color reference and a key for their lengths. Although the website provides multiple features, such as a grid format, the platform's basic form without any grid will be used for the task sequences. The rods must be presented without numerical values so that each bar can have any size unit determined in the task question.

Figure 3

Mathsbot Application: Rods



Task Sequences

The task sequences shared in this article are designed to progress from building the meaning of fraction-as-a-measure to more sophisticated fraction concepts. The tasks are modified to facilitate whole-class instruction, which would allow for individual or small group work followed by whole-class discussion orchestrated by the teacher. The initial sequence aims to familiarize students with partitioning and iterating actions, which will lay the foundation for the rest of the task sequences. Note that only one task is provided for each type in the sequences; multiple tasks should be created for each type if students need extensions or further practice. Templates of the sequences are shared separately via a QR code at the end of this article. Potential solutions created in Mathsbot are included as figures within this article.

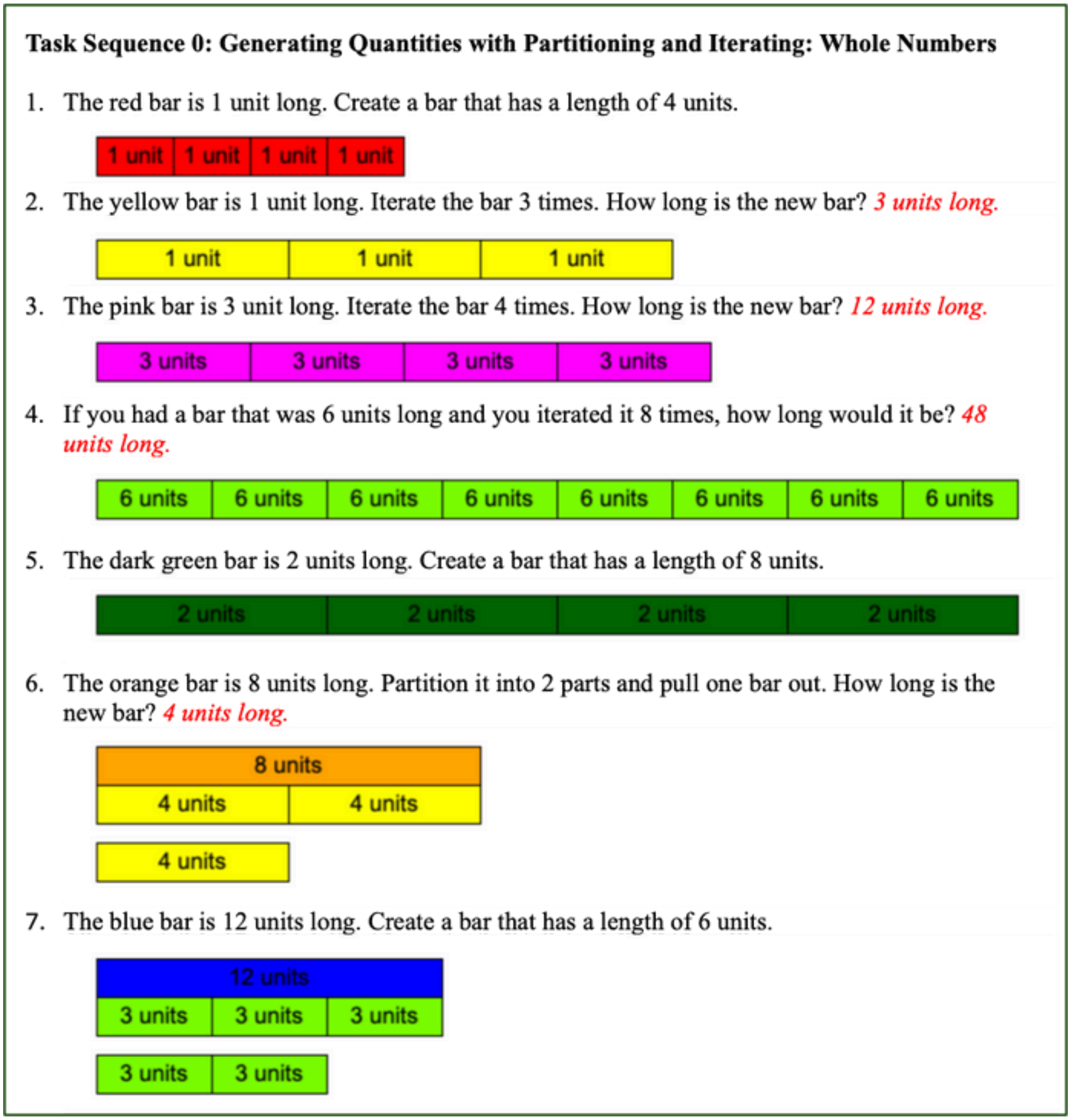
Task Sequence 0. Generating Quantities with Partitioning and Iterating: Whole Numbers

This task sequence aims to help students understand how to approach "partitioning" and "iterating" actions using Cuisenaire Rods. When partitioning a bar, students use other bars to create sub-parts. Similarly, when iterating, students use bars of the same size multiple times. Therefore, it is important to provide students with additional pieces. In virtual form, the iterating action can be conducted using the arrow pointing right (i.e., the "copy right" button), which also allows the bars to be lined up. The "new bar" can be composed of several pieces, and students can also add a bar and adjust its size by dragging it to match the result of partitioning or iterating. An example task can be completed first with the whole class so students can understand the expectations and meanings of terms like partition, iterate, and new bar.

Fractions for All *(continued)*

Figure 4

Solutions for Task Sequence 0



Task Sequence 0: Generating Quantities with Partitioning and Iterating: Whole Numbers

1. The red bar is 1 unit long. Create a bar that has a length of 4 units.

2. The yellow bar is 1 unit long. Iterate the bar 3 times. How long is the new bar? *3 units long.*

3. The pink bar is 3 unit long. Iterate the bar 4 times. How long is the new bar? *12 units long.*

4. If you had a bar that was 6 units long and you iterated it 8 times, how long would it be? *48 units long.*

5. The dark green bar is 2 units long. Create a bar that has a length of 8 units.

6. The orange bar is 8 units long. Partition it into 2 parts and pull one bar out. How long is the new bar? *4 units long.*

7. The blue bar is 12 units long. Create a bar that has a length of 6 units.

Figure 4 illustrates the task sequence with example solutions that were created in Mathsbot. The labels for the bars are added to the solutions for further description and may not be necessary in the solutions created by students. For Tasks 6 and 7, students are expected to decide the result of partitioning with a trial-and-error method. That means, when partitioning the orange bar into two parts, they may try several bars until they decide that the yellow bar gives them equal size partitioning. Such trial-and-error methods may be needed for the tasks in the later sequences as well.

Fractions for All *(continued)*

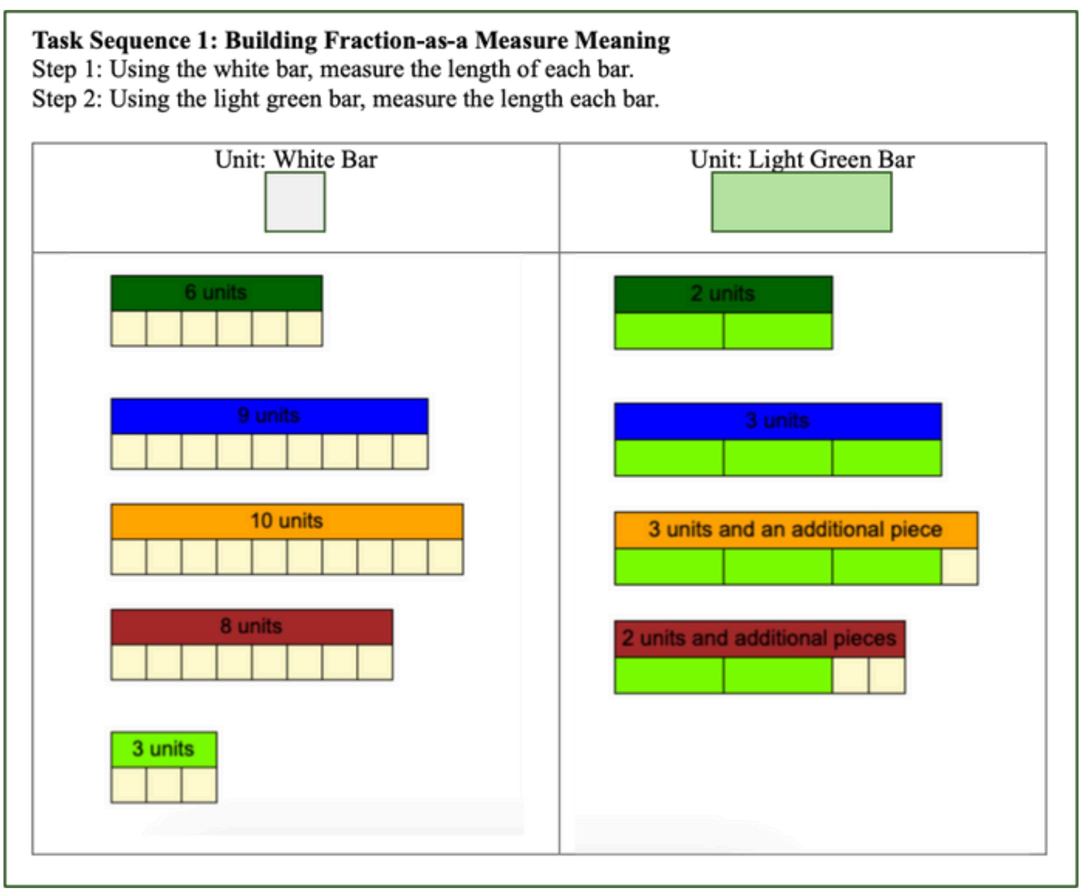
Task Sequence 1. Building Fraction-as-a-Measure Meaning

The purpose of this task is to help students understand fraction-as-a-measure. The learning goal for this task can be articulated as follows: To measure quantities smaller than a unit, fractions are necessary. The unit can be partitioned into partial units. A partial unit measures the unit a certain number of times. The size of a quantity can be specified by the number and size of a partial unit. Figure 5 includes the task sequence, as well as potential solutions that were created in Mathsbot.

In Step 1, the white bar (i.e., the unit) measures each bar a whole number of times. Students iterate the bars to measure the quantities. In Step 2, the measurement activity starts similarly to Step 1. The initial tasks of Step 2 build on the previously available activity, using the action of "iterating" to measure. Therefore, in Step 2, the quantities to be measured are still greater than the unit. The light green bar measures the dark green bar and the blue bar a whole number of times. However, when measuring the orange and brown bars, the need for a partial unit arises. Students may describe the situation as, "Three light green bars and one small additional piece measure the orange bar." When asked to describe the additional piece in terms of the light green bar, they describe it by partitioning the unit bar into sub-parts: "You will break the light green bar into three pieces and take one [or two] of the pieces." Eventually, if not already introduced, fraction language can be introduced as follows: "When a partial unit measures the unit 3 times, we call the partial 'one-third' and write it as 1/3."

Figure 5

Solutions for Task Sequence 1



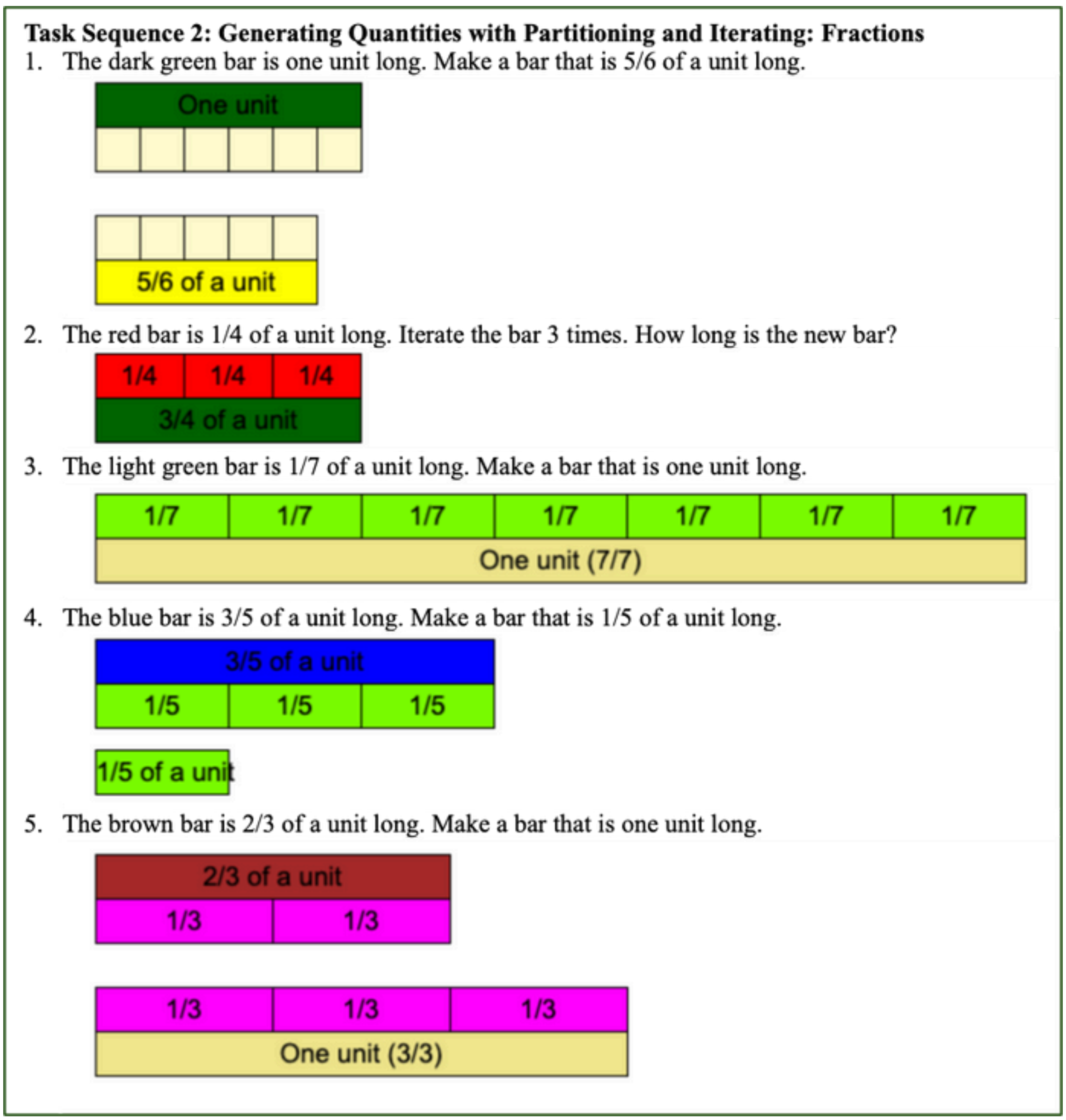
Fractions for All *(continued)*

Task Sequence 2. Generating Quantities with Partitioning and Iterating: Fractions

Building on the understanding promoted from the previous task sequences, the learning goal of this sequence is explained with the following examples: When given a bar that is one unit long and asked to create a fraction, such as 5/8, we partition the bar into 8 pieces, knowing that each piece is 1/8 of a unit. If we pull out one piece and iterate it 5 times, the new bar is 5/8 of a unit long. Conversely, when given a bar with a fraction size, such as 4/7, and asked to create a unit, we partition the bar into 4 pieces, knowing that each piece is 1/7 of a unit, and iterate the piece 7 times to obtain 7/7 of a unit, which makes a complete unit. Figure 6 includes the task sequence, as well as potential solutions that were created in Mathsbot.

Figure 6

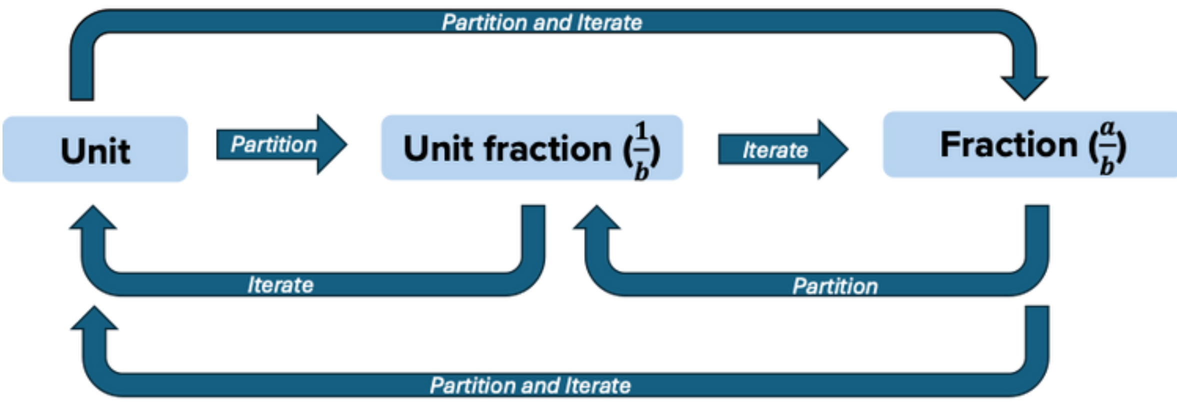
Solutions for Task Sequence



In a prior study, a similar task sequence was presented to a pre-service teacher, Nina (pseudonym). Nina successfully completed Tasks 1 and 2, which involved creating a fractional quantity from a bar that is either one unit long or a unit fraction amount. However, in Task 3, a reverse task of Task 1, when asked to create the unit from a unit fraction, 1/7, instead of iterating the unit fraction, she partitioned the bar into 7 parts. When asked "1/7 of what [unit]?", Nina realized her mistake and correctly iterated the given bar to obtain the whole unit. Nina's initial response to Task 3 highlighted a common struggle with reverse tasks. Similarly, when asked to create the unit from a bar with a non-unit fraction (i.e., Task 5 type tasks), she again encountered difficulty in successfully solving the tasks.

Figure 7

Analysis Chart for the Task Sequence 2



Nina's struggle indicated a misalignment between the original task sequence and her existing understanding, suggesting that the original task sequence was not effectively building upon her current knowledge. This led to further analysis of the task sequence illustrated in Figure 7. Tasks 1 and 2 are represented with arrows pointing to the right, indicating forward reasoning. Task 3, requiring reverse reasoning and iterating action, is represented with an arrow from a unit fraction to the unit pointing left. Task 4, requiring reverse reasoning and partitioning a proper fraction to obtain a unit fraction, is represented with an arrow from the proper fraction to the unit fraction pointing left. Finally, Task 5 is represented with an arrow from the proper fraction to the unit pointing left. The analysis results demonstrated that supporting Task 4 reasoning was essential in enabling understanding for Task 5. Task 4 reasoning involves a partitioning action while considering the size of the bar as a collection of unit fractions. Including such tasks that focus on creating a unit fraction from a bar with a measure of a/b helped Nina develop reverse reasoning. The conclusion from these findings was that when designing task sequences, elements of reverse reasoning tasks should be incorporated, as Nina's struggle demonstrated that these tasks are crucial for building on students' existing knowledge and fostering deeper conceptual understanding.

Closing Thoughts

The task sequences described here support students in constructing their own understanding of fraction concepts by completing the task sequences and engaging in both mental and physical activities required in the tasks. It is crucial that students progress successfully from one type of task to the next. If students struggle to reason correctly in a particular task, it indicates they may lack the prerequisite understanding. In such cases, previous

Fractions for All *(continued)*

tasks should be revisited. This process is about supporting students in building their understanding independently, instead of telling them what to do next. This approach accommodates students with varying levels of initial knowledge, thus allowing them to learn at their own pace and according to their individual needs.

Resources*: Task Sequences: https://tinyurl.com/4x3jrcjh*

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